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Regular languages of plus- and minus-(in)decomposable permutations

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Abstract. In this paper we show that the sets of plus- respectively minus-decomposable permutations of a regular class under the rank encoding form regular sublanguages. Plus- respectively minus-decomposable permutations are permutations allowing for a special kind of block-decomposition. The language theoretic results shown have different approaches, even though the properties of plus- and minus-decomposability of permutations are similar.

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1 Introduction

Our interest lies in regular language and automata theoretic questions related to permutation pattern classes, i.e. downwards closed sets of permutations under the classical partial order of pattern involvement [5]. Research in languages of pattern classes is scarce, our motivation to work in this branch of permutation pattern classes is that it provides a highly computable and convenient representation of some pattern classes. We will identify these pattern classes with the languages they give rise to under the rank encoding, a form of Lehmer code [9], which is described in detail in [2, 4, 6]. The *rank encoding* of a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ is the sequence $E(\pi) = p_1 p_2 \dots p_n$ where for all $i \in \{1, \dots, n\}$,

$$p_i = \left|\left\{x : x \in \{\pi(i), \pi(i+1), \dots, \pi(n)\}, x \leq \pi(i)\right\}\right| ,$$

is called the *rank* of $\pi(i)$, amongst the entries of π that have not occurred yet.

It is important to note for the characterisation of the language of rank encoded permutations, that not every sequence in $\{1, \dots, n\}^n$ represents a permutation, as there are more sequences in $\{1, \dots, n\}^n$

than permutations of length n. For example the sequence 334644211 is the rank encoding of the permutation 346978215, whereas the sequence 234664311 cannot be decoded into a permutation. Additionally, we can see that the rank encoding is unique to every permutation and vice versa.

Following [2], the class of permutations with maximum rank $k \in \mathbb{N}$, will be denoted as Ω_k. In [2] Albert, Atkinson and Ruškuc determined that $E(\Omega_k)$ is a regular language.

We will call a permutation pattern class $\mathcal{C}$ a *regular pattern class* or a *regular class*, if $E(\mathcal{C})$ is a regular subset of $E(\Omega_k)$, for some finite $k \in \mathbb{N}$ being the size of the alphabet of the language.

Furthermore we need to introduce the general concept of block-decomposition of permutations, of which plus- and minus-decomposition are special cases.

An *interval* (or *block* see [1]) in a permutation σ is a set of contiguous values of σ such that their indices are consecutive. In the permutation $\pi = 346978215$ for example, $\pi(4)\pi(5)\pi(6) = 978$ is an interval, whereas $\pi(1)\pi(2)\pi(3)\pi(4) = 3469$ is not. It is easy to see that every permutation of length n has intervals of length 0, 1 and n, at least. The permutations of length n that *only* contain intervals of length 0, 1 and n are said to be *simple* [7].

Given a permutation π of length m and nonempty permutations $\alpha_1, \dots, \alpha_m$ the *inflation* of π by $\alpha_1, \dots, \alpha_m$, written as $\pi[\alpha_1, \dots, \alpha_m]$, is the permutation obtained by replacing each entry $\pi(i)$ by the interval that is order isomorphic to α_i, where the relative ordering of the intervals corresponds to the ordering of the entries of π [1]. Conversely a *block-decomposition* or *deflation* [1] of σ is any expression of σ as an inflation $\sigma = \pi[\alpha_1, \dots, \alpha_m]$.

For example the inflation of $\pi = 24513$ with $\alpha_1 = 12,\ \alpha_2 = 1,\ \alpha_3 = 312,\ \alpha_4 = 21,\ \alpha_5 = 1$ is $24513[12, 1, 312, 21, 1] = 346978215$, and in other words a possible block-decomposition of 346978215 is $24513[12, 1, 312, 21, 1]$. This inflation and decomposition is shown in Figure 1.

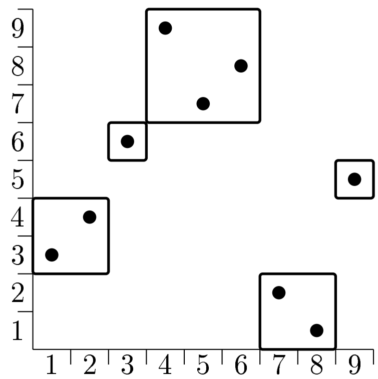

Figure 1: Plot of inflation $24513[12, 1, 312, 21, 1] = 346978215$.

This decomposition is not unique for σ. Albert and Atkinson proved:

PROPOSITION 1.1 ([1]) *Let σ be a permutation of finite length greater than 1. There is a unique simple finite permutation π, with $|\pi| > 1$ and a sequence $\alpha_1, \dots, \alpha_n$ of non-empty permutations such that*

$$\sigma = \pi[\alpha_1, \dots, \alpha_n].$$

If $\pi \neq 12, 21$ then $\alpha_1, \dots, \alpha_n$ are also uniquely determined by σ. If $\pi = 12$ or 21, then α_1, α_2 are unique so long as we require that α_1 is plus-indecomposable or minus-indecomposable respectively.

Plus- and minus-(in)decomposability are defined in Sections 2 and 3 respectively.

Utilising the above proposition, the unique block-decomposition of 346978215 with a simple permutation is given by $2413[12, 1423, 21, 1]$ as shown in Figure 2. As an example for why we have to specify additional conditions for $\pi = 12$ or 21, let $\sigma = 123456$ and $\pi = 12$, then a possible decomposition is $12[1, 12345]$, another is $12[12, 1234]$. Similarly for $\pi = 21$, take for example $\sigma = 654321$ then two possible decompositions are $21[321, 321]$ and $21[54321, 1]$.

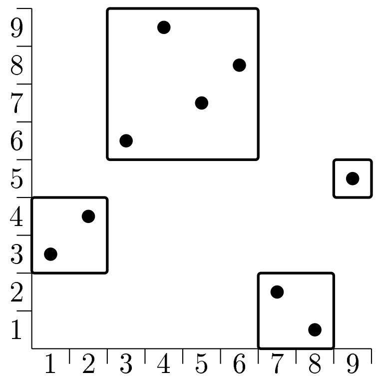

Figure 2: Plot of unique block-decomposition of $346978215 = 2413[12, 1423, 21, 1]$ as defined by Proposition 1.1, where 2413 is simple.

Over the course of this paper we will be proving the following main theorem:

THEOREM 1.2 *Let $\mathcal{C}$ be a regular pattern class. The following languages are all regular languages:*

(a) *The rank encodings of all plus-decomposable permutations in $\mathcal{C}$;*

(b) *the rank encodings of all plus-indecomposable permutations in $\mathcal{C}$;*

(c) *the rank encodings of all minus-decomposable permutations in $\mathcal{C}$;*

(d) *the rank encodings of all minus-indecomposable permutations in $\mathcal{C}$.*

Part (a) and (b) of this Theorem are proved in Section 2 as Corollary 2.3 and Theorem 2.2. Part (c) and (d) are proved in Section 3 as Theorem 3.1 and Corollary 3.2.

2 Plus-Decomposable and -Indecomposable Permutations

One of the special cases in Proposition 1.1 is the block-decomposition with $\pi = 12$. A permutation σ is said to be *plus-decomposable* (or is a *direct sum* of α_1 and α_2) if it can be written in the block-decomposition as

$$\sigma = 12[\alpha_1, \alpha_2].$$

Conversely, we call a permutation *plus-indecomposable* if it has no plus-decomposition [1].

In general, α_1 and α_2 are not unique, but if we require α_1 to be plus-indecomposable, both α_1 and α_2 are unique to σ.

The following remark is a characterisation of plus-decomposable permutations, to outline the form these permutations have. It additionally facilitates the characterisation of plus-indecomposable permutations.

Remark 2.1 Let $\sigma = \sigma(1)\dots\sigma(n)$ be a permutation, then the following are equivalent:

- $\sigma = 12[\alpha_1, \alpha_2]$ is plus-decomposable with $|\alpha_1| = \ell$ and $|\alpha_2| = n - \ell$ for $\ell \in \mathbb{N} \setminus \{0\},\ \ell < n$.
- $E(\sigma) = E\big(\sigma(1)\dots\sigma(\ell)\ \sigma(\ell+1)\dots\sigma(n)\big) =$ $= E\big(\sigma(1)\dots\sigma(\ell)\big)\ E\big(\sigma(\ell+1)\dots\sigma(n)\big) = E(\alpha_1)E(\alpha_2)$.
- $\sigma = \eta\tau$, where $\eta = \sigma(1)\dots\sigma(\ell) \in S_\ell$ is a permutation of length ℓ and τ is a permutation of $\{\ell+1, \dots, n\}$.

To prove that the set of plus-decomposable permutations and the set of plus-indecomposable permutations of a regular pattern class $\mathcal{C}$ are regular under the rank encoding, we will first prove that the subset of plus-indecomposable permutations of a regular pattern class $\mathcal{C}$ form a regular language under the rank encoding. Then it will follow that the complement set of plus-decomposable rank encoded permutations is also regular, since the family of regular languages are closed under complement. The proof considers the automaton that accepts the regular class $\mathcal{C}$ and modifies it to accept only plus-indecomposable rank encoded permutations.

Theorem 2.2 (Theorem 1.2 part (b)) *Let $\mathcal{C}$ be a regular class. Then $\mathcal{I}_P(\mathcal{C})$, the set of all plus-indecomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.*

Proof. Conversely to the characterisation in Remark 2.1, the rank encoding $E(\pi)$ of a plus-indecomposable permutation π of length n never contains an initial segment of the form $E(\pi(1)\dots\pi(\ell)) = E(p)$ where $\ell < n$ and p is a permutation of length ℓ.

We will utilise this description to construct the automaton accepting the language of the set of plus-indecomposable permutations under the rank encoding.

The automaton accepting $E(\mathcal{I}_P(\mathcal{C}))$ is based on the unique minimal automaton of $E(\mathcal{C})$. Let that automaton be

$$\mathscr{A} = \big(\Sigma,\ S,\ a,\ F,\ t\big),$$

where Σ is the alphabet, S the set of states, a the start state, F the set of accept states and $t\colon S \times \Sigma \to S$ the transition function.

We will construct the automaton accepting only the plus-indecomposable rank encoded permutations as follows

$$\mathscr{I}_P = \big(\Sigma,\ S \cup \{x, y\},\ x,\ F \cup \{x\},\ t'\big),$$

where x and y are new states and $t'\colon (S \cup \{x, y\}) \times \Sigma \to S \cup \{x, y\}$ is a new transition function defined as:

$$\begin{aligned} t'(y, \alpha) &= y \\ t'(x, \alpha) &= t(a, \alpha) \\ t'(s_0, \alpha) &= y \\ t'(s, \alpha) &= t(s, \alpha) \end{aligned}$$

for all $\alpha \in \Sigma$; $s \in S \setminus F$; $s_0 \in F$.

In $\mathscr{I}_P$ the new states x, y are the new start state and sink state respectively. A sink state is a state q where q is not the start state and $q \notin F$, additionally the transition t from q for any letter $\alpha \in \Sigma$ is $t(q, \alpha) = q$. We are introducing a new start state, to avoid the resulting language being empty, in case the original start state a is an accept state.

Let us now show that the new automaton $\mathscr{I}_P$ indeed only accepts the rank encodings corresponding to plus-indecomposable permutations from the automaton $\mathscr{A}$ of the regular class $\mathcal{C}$. Let w be a word accepted by $\mathscr{A}$, if we end up in an accept state of $\mathscr{A}$ before reading the entire word, so there is an initial segment in w that is a rank encoding of a permutation in $\mathcal{C}$, then the new transition function t' will send us to the sink state y and the word w will not be accepted by $\mathscr{I}_P$. On the other hand if w is a word accepted by $\mathscr{A}$ and by $\mathscr{I}_P$, we end up in an accept state only when the entire word w is read. The first case is only possible for plus-decomposable permutations, as shown in Remark 2.1.

The automaton $\mathscr{I}_P$ only accepts the words corresponding to plus-indecomposable permutations of $\mathcal{C}$ under the rank encoding. Thus the language $E(\mathcal{I}_P(\mathcal{C}))$ is regular, which concludes the proof of Theorem 2.2. □

Corollary 2.3 (Theorem 1.2 part (a)) *Let $\mathcal{C}$ be a regular class. Then $\mathcal{D}_P(\mathcal{C})$, the set of all plus-decomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.*

Proof. Let $\mathcal{I}_P(\mathcal{C}) \subseteq \mathcal{C}$ be the set of all plus-indecomposable permutations in $\mathcal{C}$.

As $\mathcal{D}_P(\mathcal{C})$ and $\mathcal{I}_P(\mathcal{C})$ are complementary in $\mathcal{C}$ we have

$$\mathcal{D}_P(\mathcal{C}) = \mathcal{C} \backslash \mathcal{I}_P(\mathcal{C}) \Rightarrow E(\mathcal{D}_P(\mathcal{C})) = E(\mathcal{C}) \cap E(\mathcal{I}_P(\mathcal{C}))^C.$$

As regular languages are closed under intersection and complement, $E(\mathcal{D}_P(\mathcal{C}))$ is regular. □

3 Minus-Decomposable and -Indecomposable Permutations

The other special case in Proposition 1.1 is the block-decomposition with $\pi = 21$. We say that a permutation σ is *minus-decomposable* (or is a *skew sum* of α_1 and α_2) if it can be written in the block-decomposition as

$$\sigma = 21[\alpha_1, \alpha_2].$$

Conversely we say that a permutation is *minus-indecomposable* if it has no minus-decomposition. The decomposition of a minus-decomposable permutation is unique, if α_1 is assumed to be minus-indecomposable.

In a regular pattern class it is simpler to deal with the language that describes the rank encoding of minus-decomposable permutations rather than the minus-indecomposable permutations.

Theorem 3.1 (Theorem 1.2 part (c)) *Let $\mathcal{C}$ be a regular class. Then $\mathcal{D}_M(\mathcal{C})$, the set of all minus-decomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.*

Proof. Let $E(\mathcal{C})$ be the regular language of $\mathcal{C}$ under the rank encoding where the alphabet of $E(\mathcal{C})$ is $\{1, \ldots, k\}$, $k \in \mathbb{N}$ and let $E(\mathcal{D}_M(\mathcal{C}))$ be the rank encoded language of $\mathcal{D}_M(\mathcal{C})$.

Let $\pi \in \mathcal{D}_M(\mathcal{C})$ be arbitrary with $\pi = 21[\alpha_1, \alpha_2]$, where $|\pi| = n$ and $|\alpha_2| = d < k$. We know that $d < k$ as otherwise the rank of the elements in π corresponding to α_1 will exceed k, contradicting that $\mathcal{D}_M(\mathcal{C}) \subseteq \mathcal{C} \subseteq \Omega_k$.

Then from Figure 3 we see that

$$E(\pi) = E\big(\pi(1) \dots \pi(n-d)\ \pi(n-d+1) \dots \pi(n)\big) = p_1 \dots p_{n-d}\ p_{n-d+1} \dots p_n\ ,$$

where $p_i > d$ for $i \leq n-d$ and $p_i \leq d$ for $i > n-d$.

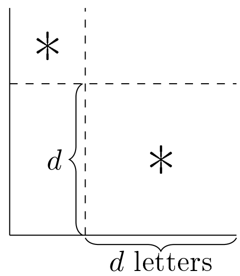

Figure 3: Plot of a minus-decomposable permutation, where $1 \leq d < k$.

In other words, for any $\pi \in \mathcal{C}$, to decide whether $\pi \in \mathcal{D}_M(\mathcal{C})$, it suffices to check whether there is an integer $d < k$, such that $E(\pi)$ consists of $n-d$ integers that are greater than d followed by d integers that are smaller or equal to d. This leads to the following languages:

$$L_d = \Big\{\{d+1, \dots, k\}^+ \{1, \dots, d\}^d\Big\} = \Big\{\{d+1, \dots, k\}\{d+1, \dots, k\}^* \{1, \dots, d\}^d\Big\}\,,$$

where L_d is a superset of sequences that are of similar form to the words in $E(\mathcal{D}_M(\mathcal{C}))$, with $d \in \{1, \dots, k-1\}$ fixed.

Next, we will merge all possibilities of L_d,

$$\mathscr{L} = \bigcup_{d=1}^{k-1} L_d,$$

$\mathscr{L}$ contains all words representing k-bounded minus-decomposable permutations. Clearly $\mathscr{L}$ is regular.

Then from the above we can find that

$$E(\mathcal{D}_M(\mathcal{C})) = \mathscr{L} \cap E(\mathcal{C}),$$

which is a regular language, as by assumption $\mathcal{C}$ is a regular class with the language $E(\mathcal{C})$ and $\mathscr{L}$ is regular by construction. Thus the language $E(\mathcal{D}_M(\mathcal{C}))$ of minus-decomposable rank encoded permutations of a regular class is regular. □

Corollary 3.2 (Theorem 1.2 part (d)) *Let $\mathcal{C}$ be a regular class. Then $\mathcal{I}_M(\mathcal{C})$ the set of all minus-indecomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.*

Proof. Similar to the proof of Corollary 2.3 we obtain the above result by complementation.

Let $\mathcal{D}_M(\mathcal{C}) \subseteq \mathcal{C}$ be the regular set containing the minus-decomposable permutations. Then

$$\begin{aligned}&\mathcal{I}_M(\mathcal{C}) \cup \mathcal{D}_M(\mathcal{C}) = \mathcal{C} \;\Rightarrow\; E(\mathcal{I}_M(\mathcal{C})) \cup E(\mathcal{D}_M(\mathcal{C})) = E(\mathcal{C})\\ &\Rightarrow\; E(\mathcal{I}_M(\mathcal{C})) = E(\mathcal{D}_M(\mathcal{C}))^C \cap E(\mathcal{C}),\end{aligned}$$

where $E(\mathcal{I}_M(\mathcal{C}))$ is the language of minus-indecomposable rank encoded permutations, and it is regular as $E(\mathcal{D}_M(\mathcal{C}))$ and $E(\mathcal{C})$ are regular. □

4 Conclusion

It is interesting to see that in a language theoretic context the approach to plus- and minus-(in)decomposable permutations is different, even though the properties of these permutations are seemingly very similar. This arises from the fundamental asymmetry in the definition of the rank encoding.

The approaches used to prove the regularity of the sets of plus- and minus-(in)decomposable permutations are highly constructive. This facilitates the development of algorithms to compute these languages in regular permutation pattern classes.

Having set up this characterisation and representation of these particular subsets of regular pattern classes opens the questions whether there are further sets of permutations in these classes that are regular. The authors thank the reviewer for the suggestion of investigating the subset of a regular class $\mathcal{C}$ of the form $\{\pi[\alpha_1, \ldots, \alpha_n] : \alpha_i \in \mathcal{C},\ \text{for all } i\}$ where π is a fixed simple permutation, which is part of further research.

The regular language approach to permutation pattern classes through the rank encoding as mentioned in [2, 4, 6] and the languages described in this paper have been implemented as algorithms in the permutation pattern class GAP [8] package `PatternClass` [3].

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