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Applying a reciprocity method to count permutations avoiding two families of consecutive patterns

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Abstract. We study the generating function $\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\text{LRmin}(\sigma)}y^{1+\text{des}(\sigma)}$ where $\mathcal{NM}_n(\tau)$ is the set of permutations σ in the symmetric group S_n which have no consecutive occurrences of τ, τ is of the form $1p2\ldots(p-1)$ or $13\ldots(p-1)2p$ for some $p\geq 4$, $\text{des}(\sigma)$ is the number of descents of σ and $\text{LRmin}(\sigma)$ is the number of left-to-right minima of σ. We show that for any $p\geq 4$, this generating function is of the form $\left(\frac{1}{U_\tau(t,y)}\right)^x$ where $U_\tau(t,y)=\sum_{n\geq 0}U_{\tau,n}(y)\frac{t^n}{n!}$ and the coefficients $U_{\tau,n}(y)$ satisfy some simple recursions depending on p.

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1 Introduction

Given a sequence $\sigma=\sigma_1\ldots\sigma_n$ of distinct integers, let the *reduction* of σ, $\text{red}(\sigma)$, be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau=\tau_1\ldots\tau_p$ in the symmetric group S_p, we say a permutation $\sigma=\sigma_1\ldots\sigma_n\in S_n$ has a *τ-match starting at position i* provided $\text{red}(\sigma_i\ldots\sigma_{i+p-1})=\tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. Given a permutation $\sigma=\sigma_1\ldots\sigma_n\in S_n$, we let $\text{des}(\sigma)=|\{i:\sigma_i>\sigma_{i+1}\}|$. We say that σ_j is a *left-to-right minimum* of σ if $\sigma_j<\sigma_i$ for all $i<j$. We let $\text{LRmin}(\sigma)$ denote the number of left-to-right minima of σ.

The main object of study in this paper is the generating function

$$NM_\tau(t,x,y)=\sum_{n\geq 0}NM_{\tau,n}(x,y)\frac{t^n}{n!} \tag{1}$$

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where $\mathcal{NM}_n(\tau)$ is the set of permutations in S_n with no τ-matches and

$$NM_{\tau,n}(x,y) = \sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\mathrm{LRmin}(\sigma)} y^{1+\mathrm{des}(\sigma)}. \tag{2}$$

In particular, the main goal of this paper is to compute the generating function $NM_\tau(t,x,y)$ and the polynomials $NM_{\tau,n}(x,y)$ for two infinite families of permutations, namely, τ of the form $1p23\ldots(p-1)$ and τ of the form $13\ldots(p-1)2p$ where $p \geq 4$. There are a number of methods that have appeared in the literature to study the generating functions for either the distribution of τ-matches in S_n, see [9, 5, 20, 24, 16], as well as methods to find the number of permutations of S_n with no τ-matches, see [6, 1, 17, 15]. None of these approaches tries to study the refined generating function $NM_{\tau,n}(x,y)$. Instead, we shall use the so-called reciprocity method introduced by the authors in [14] to compute generating functions of the form $NM_\tau(t,x,y)$ where τ is a permutation which starts with 1. In particular, the authors [14] proved that in such a situation, one can always write the generating function $NM_\tau(t,x,y)$ as

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}. \tag{3}$$

Thus

$$U_\tau(t,y) = \frac{1}{1+\sum_{n\geq 1} NM_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{4}$$

One can then use the homomorphism method to give a combinatorial interpretation the right-hand side of (4) which can be used to find simple recursions for the coefficients $U_{\tau,n}(y)$. The homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{5}$$

where $H(t)$ and $E(t)$ are the generating functions for the homogeneous and elementary symmetric functions, respectively:

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} \frac{1}{1-x_i t}, \quad E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t. \tag{6}$$

See, for example, [2, 18, 19, 20, 21, 22, 23]. In our case, we define a homomorphism θ on Λ by setting

$$\theta(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1,y).$$

Then

$$\theta(E(-t)) = \sum_{n\geq 0} NM_{\tau,n}(1,y)\frac{t^n}{n!} = \frac{1}{U_\tau(t,y)}.$$

Hence

$$U_\tau(t,y) = \frac{1}{\theta(E(-t))} = \theta(H(t))$$

which implies that

$$n!\theta(h_n) = U_{\tau,n}(y). \tag{7}$$

Thus if we can compute $n!\theta(h_n)$ for all $n \geq 1$, then we can compute the polynomials $U_{\tau,n}(y)$ and the generating function $U_\tau(t,y)$ which in turn allows us to compute the generating function $NM_\tau(t,x,y)$.

In [14], the authors studied the generating functions $U_\tau(t,y)$ for permutations τ of the form $\tau = 1324\ldots p$ where $p \geq 4$. That is, τ arises from the identity permutation by transposing 2 and 3. Using the homomorphism method, the authors [14] proved that $U_{1324,1}(y) = -y$ and for $n \geq 2$,

$$U_{1324,n}(y) = (1-y)U_{1324,n-1}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{1324,n-2k+1}(y) \tag{8}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number. They also proved that for any $p \geq 5$, $U_{1324\ldots p,n}(y) = -y$ and for $n \geq 2$,

$$U_{1324\ldots p,n}(y) = (1-y)U_{1324\ldots p,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor + 1} (-y)^{k-1} U_{1324\ldots p,n-((k-1)(p-2)+1)}(y). \tag{9}$$

The main goal of this paper is to prove the following two theorems.

THEOREM 1.1 *Let* $\tau = 1p23\ldots(p-1)$ *where* $p \geq 4$. *Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \quad \textit{where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, *and, for* $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor + 1} (-y)^{k-1} \binom{n-(k-1)(p-3)-2}{k-1} U_{\tau,n-((k-1)(p-2)+1)}(y).$$

We note that the special case of Theorem 1.1 where $p = 4$ was proved in the extended abstract [13].

THEOREM 1.2 *Let* $\tau = 13\ldots(p-1)2p$ *where* $p \geq 4$. *Then*

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \quad \textit{where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y) = -y$, *and, for* $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)^k \frac{1}{(p-2)k+1}\binom{k(p-1)}{k} U_{\tau,n-(k(p-2)+1)}(y). \tag{10}$$

For $p \geq 5$, these two recursions are more complicated than the recursion for τ of the form $1324\ldots p$ given in (9) in that the recursions for τ of the form $1p23\ldots(p-1)$ involve binomial coefficients and the recursions for τ of the form $13\ldots(p-1)2p$ involve coefficients which count the number of $(p-1)$-ary trees. In all three cases described above, computational evidence suggests that the polynomials $U_{\tau,n}(-y)$ are log-concave polynomials. In the case where $p=3$, the permutation $1p2\ldots(p-1)$ becomes 132 and the permutation $13\ldots(p-1)2p$ becomes 123. The authors computed explicit formulas for $NM_{132}(t,x,y)$ and $NM_{123}(t,x,y)$ using other methods in [12].

The outline of this paper is as follows. In Section 2, we recall the background in the theory of symmetric functions that we will need for our proofs. Then in Section 3, we prove Theorems 1.1 and 1.2, Finally in Section 4, we state our conclusions and discuss some areas for further research.

2 Symmetric functions

In this section, we give the necessary background on symmetric functions that will be needed for our proofs.

A partition of a positive integer n is a vector of non-zero integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ where $0 < \lambda_1 \leq \cdots \leq \lambda_s$ and $n = \lambda_1 + \ldots + \lambda_s$. Each λ_i for $1 \leq i \leq s$ is called a *part* of λ and we let $\ell(\lambda)$ denote the number of parts of λ. We use the notation $\lambda \vdash n$ to mean λ is a partition of n. When a partition of n involves repeated parts, we shall often use exponents in the partition notation to indicate these repeated parts. For example, we will write $(1^2, 2^3, 3^2)$ for the partition $(1,1,2,2,2,3,3)$.

Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. The n^{th} elementary symmetric function $e_n = e_n(x_1, x_2, \ldots)$ and n^{th} homogeneous symmetric function $h_n = h_n(x_1, x_2, \ldots)$ are defined by the generating functions given in (6). For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. It is well known that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ. In particular, $e_0, e_1, \ldots$ is an algebraically independent set of generators for Λ and, hence, a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n.

A key element of our proofs is the combinatorial description of the coefficients of the expansion of h_n in terms of the elementary symmetric functions e_λ given by Eğecioğlu and the second author in [7]. They defined a λ-brick tabloid of shape (n) with $\lambda \vdash n$ to be a rectangle of height 1 and length n which is covered by "bricks" of lengths found in the partition λ in such a way that no two bricks overlap. For example, Figure 1 shows the six $(1^2, 2^2)$-brick tabloids of shape (6).

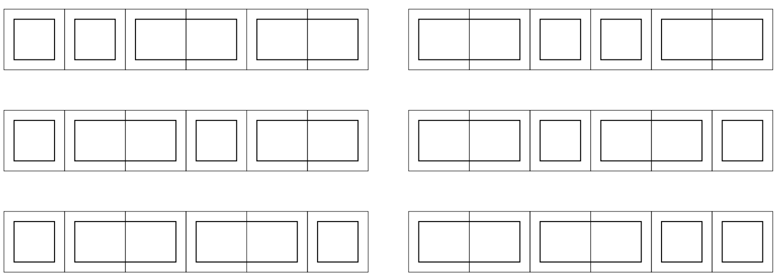

Figure 1: All six $(1^2, 2^2)$-brick tabloids of shape (6).

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ-brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ-brick tabloids of shape (n). If $B \in \mathcal{B}_{\lambda,n}$, we will write $B = (b_1, \ldots, b_{\ell(\lambda)})$ if the lengths of the bricks in B, reading from left to right, are $b_1, \ldots, b_{\ell(\lambda)}$. Through simple recursions, Eğecioğlu and the second author [7] proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n}\ e_\lambda. \tag{11}$$

This interpretation of h_n in terms of e_n will aid us in describing the coefficients of $\theta(H(t)) = U_\tau(t,y)$ which will in turn allow us to compute the coefficients $NM_{\tau,n}(x,y)$.

3 The proof of Theorems 1.1 and 1.2.

3.1 The homomorphism method and an involution

First we recall the key steps in the required application of the homomorphism method for our problem as described in [14]. Suppose that $\tau \in S_j$ is a permutation such that τ starts with 1 and $\mathrm{des}(\tau) = 1$. Our first step is to give a combinatorial interpretation to

$$U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)} = \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} NM_{\tau,n}(1,y)} \tag{12}$$

where $NM_{\tau,n}(1,y) = \sum_{\sigma \in \mathcal{NM}_n(\tau)} y^{1+\mathrm{des}(\sigma)}$.

Following [14], we define a ring homomorphism θ_τ on the ring of symmetric functions Λ by setting $\theta_\tau(e_0) = 1$ and

$$\theta_\tau(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1,y) \text{ for } n \geq 1. \tag{13}$$

It follows that

$$\begin{aligned} \theta_\tau(H(t)) &= \sum_{n \geq 0} \theta_\tau(h_n) t^n = \frac{1}{\theta_\tau(E(-t))} = \frac{1}{1 + \sum_{n \geq 1} (-t)^n \theta_\tau(e_n)} \\ &= \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} NM_{\tau,n}(1,y)} = U_\tau(t,y) \end{aligned}$$

which is what we want to compute.

By (11), we have that

$$\begin{aligned} n!\theta_\tau(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n}\ \theta_\tau(e_\lambda) \\ &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i}}{b_i!} NM_{\tau,b_i}(1,y) \\ &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \binom{n}{b_1, \ldots, b_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} NM_{\tau,b_i}(1,y). \end{aligned} \tag{14}$$

Our next goal is to give a combinatorial interpretation to the right-hand side of (14). Fix a partition λ of n and a λ-brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$. Then we can interpret $\binom{n}{b_1,\ldots,b_{\ell(\lambda)}}$ as the number of ways of picking an ordered set partition $(S_1, \ldots, S_{\ell(\lambda)})$ of $\{1, \ldots, n\}$ such that $|S_i| = b_i$ for $i = 1, \ldots, \ell(\lambda)$. We will interpret $\prod_{i=1}^{\ell(\lambda)} NM_{\tau,b_i}(1, y)$ as the number of ways of picking permutations $(\sigma^{(1)}, \ldots, \sigma^{(\ell(\lambda))})$ such that $\sigma^{(i)} \in \mathcal{NM}_{b_i}(\tau)$ and assigning a weight to this $\ell(\lambda)$-tuple to be $\prod_{i=1}^{\ell(\lambda)} y^{\text{des}(\sigma^{(i)})+1}$.

We can then use the triple $\langle B, (S_1, \ldots, S_{\ell(\lambda)}), (\sigma^{(1)}, \ldots, \sigma^{(\ell(\lambda))})\rangle$ to construct a filled-labeled-brick tabloid $O_{\langle B,(S_1,\ldots,S_{\ell(\lambda)}),(\sigma^{(1)},\ldots,\sigma^{(\ell(\lambda))})\rangle}$ as follows. First for each brick b_i, we place a permutation $\tau^{(i)}$ of S_i in the cells of the brick, reading from left to right, so that $\text{red}(\tau^{(i)}) = \sigma^{(i)}$. Then we label each cell of b_i that starts a descent of $\tau^{(i)}$ with a y and we also label the last cell of b_i with y. This accounts for the factor $y^{\text{des}(\sigma^{(i)})+1}$. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from y to $-y$. For example, suppose $n = 19$, $\tau = 15234$, $B = (9, 3, 5, 2)$, $S_1 = \{2, 5, 6, 9, 11, 15, 16, 17, 19\}$, $S_2 = \{7, 8, 14\}$, $S_3 = \{1, 3, 10, 13, 18\}$, $S_4 = \{4, 12\}$, $\sigma^{(1)} = 1\ 6\ 2\ 9\ 7\ 8\ 3\ 4\ 5 \in NM_{\tau,9}$, $\sigma^{(2)} = 1\ 3\ 2 \in NM_{\tau,3}$, $\sigma^{(3)} = 5\ 1\ 2\ 4\ 3 \in NM_{\tau,5}$, and $\sigma^{(4)} = 2\ 1 \in NM_{\tau,2}$. Then we have pictured the filled-labeled brick tabloid $O_{\langle B,(S_1,\ldots,S_4),(\sigma^{(1)},\ldots,\sigma^{(4)})\rangle}$ constructed from the triple $\langle B, (S_1, \ldots, S_4), (\sigma^{(1)}, \ldots, \sigma^{(4)})\rangle$ in Figure 2.

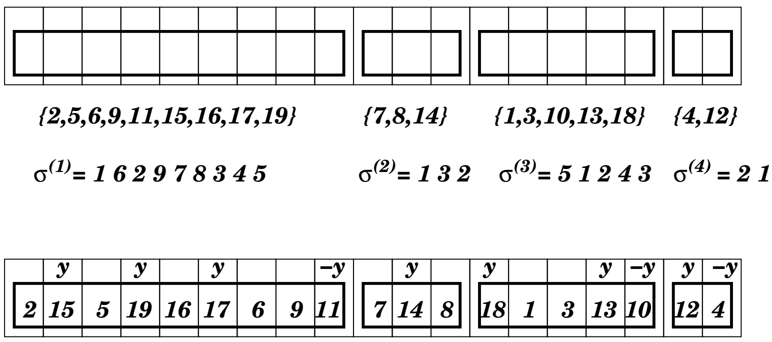

Figure 2: The construction of a filled-labeled-brick tabloid.

Given $O = O_{\langle B,(S_1,\ldots,S_{\ell(\lambda)}),(\sigma^{(1)},\ldots,\sigma^{(\ell(\lambda))})\rangle}v$, let σ be the permutation which is obtained by reading the elements in the cells of O from left to right. Then it is easy to see that we can recover O and the labels on the cells of O from B and σ. Thus we shall specify the filled-labeled-brick tabloid $O_{\langle B,(S_1,\ldots,S_{\ell(\lambda)}),(\sigma^{(1)},\ldots,\sigma^{(\ell(\lambda))})\rangle}$ by (B, σ). We let $\mathcal{O}_{\tau,n}$ denote the set of all filled-labeled-brick tabloids constructed in this way. That is, $\mathcal{O}_{\tau,n}$ consists of all pairs $O = (B, \sigma)$ where

1. $B = (b_1, \ldots, b_{\ell(\lambda)})$ is brick tabloid of shape (n),
2. $\sigma \in S_n$ such that there is no τ-match of σ which is entirely contained in a single brick of B, and
3. if there is a cell c such that a brick b_i contains both cells c and $c + 1$ and $\sigma_c > \sigma_{c+1}$, then cell c is labeled with a y and the last cell of any brick is labeled with $-y$.

The sign of O, $\text{sgn}(O)$, is $(-1)^{\ell(\lambda)}$ and the weight of O, $W(O)$, is $y^{\ell(\lambda)+\text{intdes}(\sigma)}$ where $\text{intdes}(\sigma)$ denotes the number of i such that $\sigma_i > \sigma_{i+1}$ and σ_i and σ_{i+1} lie in the same brick. We shall refer to such i as an *internal descent* of O. For example, if $\tau = 15234$, then such a filled-labeled-brick tabloid O constructed from the brick tabloid $B = (2, 8, 3)$ is pictured in Figure 3 where $W(O) = y^7$ and

$\text{sgn}(O) = (-1)^3$. Note that the labels on O are completely determined by the underlying brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$ and the underlying permutation σ. Thus the filled-labeled-brick tabloid O pictured in Figure 3 equals $((2,8,3), 4\ 11\ 7\ 8\ 10\ 5\ 12\ 3\ 9\ 6\ 2\ 1\ 13)$.

It follows that

$$n!\theta_\tau(h_n) = \sum_{O \in \mathcal{O}_{\tau,n}} sgn(O) W(O). \tag{15}$$

	−y			y		y		y	−y	y		−y
4	11	7	8	10	5	12	3	9	6	2	1	13

Figure 3: An element of $\mathcal{O}_{15234,13}$.

We shall often want to start with a filled-labeled-brick tabloid $O = (B, \sigma)$ and remove the first k cells of O and consider the resulting object $\text{red}_k(B, \sigma) = (B', \alpha)$ where B' is the brick tabloid whose bricks end at those cells $c > k$ where cell c is the end of a brick in B and whose permutation α is $\text{red}(\sigma_{k+1} \ldots \sigma_n)$. For example, if O is the filled-labeled-brick tabloid pictured in Figure 3, then $\text{red}_4(O)$ is pictured in Figure 4.

y		y		y	−y	y		−y
7	4	8	3	6	5	2	1	9

Figure 4: $\text{red}_4(O)$ for O in Figure 3.

Next we define a weight-preserving sign-reversing involution I_τ on $\mathcal{O}_{\tau,n}$. Given an element $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \ldots \sigma_n$, scan the cells of O from left to right looking for the first cell c such that either

(i) c is labeled with a y or

(ii) c is a cell at the end of a brick b_i, $\sigma_c > \sigma_{c+1}$, and there is no τ-match of σ that lies entirely in the cells of bricks b_i and b_{i+1}.

In case (i), if c is a cell in brick b_j, then we split b_j into two bricks b'_j and b''_j where b'_j contains all the cells of b_j up to an including cell c and b''_j consists of the remaining cells of b_j and we change the label on cell c from y to $-y$. In case (ii), we combine the two bricks b_i and b_{i+1} into a single brick b and change the label on cell c from $-y$ to y. For example, consider the element $O \in \mathcal{O}_{13245,13}$ pictured in Figure 3. Note that even though the number in the last cell of brick 1 is greater than the the number in the first cell of brick 2, we can not combine these two bricks because 4 11 7 8 10 would be a 15234-match. Thus the first place that we can apply the involution is on cell 5 which is labeled with a y so that $I_\tau(O)$ is the object pictured in Figure 5. Finally, if neither case (i) or case (ii) applies, then we define $I_\tau(O) = O$.

	−y			−y		y		y	−y	y		−y
4	11	7	8	10	5	12	3	9	6	2	1	13

Figure 5: $I_\tau(O)$ for O in Figure 3.

In [14], the authors proved that I is an involution if τ starts with 1 and $\mathrm{des}(\tau) = 1$. It is clear from our definitions that if $I_\tau(O) \neq O$, then $\mathrm{sgn}(O)W(O) = -\mathrm{sgn}(I_\tau(O))W(I_\tau(O))$. Hence it follows from (15) that

$$n!\theta_\tau(h_n) = \sum_{O \in \mathcal{O}_{\tau,n}} \mathrm{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} \mathrm{sgn}(O)W(O). \tag{16}$$

Hence if τ starts with 1 and $\mathrm{des}(\tau) = 1$, then

$$U_{\tau,n}(y) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} \mathrm{sgn}(O)W(O). \tag{17}$$

Thus to compute $U_{\tau,n}(y)$, we must analyze the fixed points of I_τ.

Note that if O is a fixed point of I_τ, then we can not apply case (i) of I_τ so that there can be no cells labeled with y which means that the elements in each brick of O must be increasing. Similarly, we cannot apply case (ii) of I_τ so that if b_i and b_{i+1} are two consecutive bricks in O, then either there is an increase between bricks b_i and b_{i+1}, i.e. the last element in b_i is less than the first element of b_{i+1}, or there is τ-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve both the last element in b_i and the first element of b_{i+1}. In addition, the authors proved in [14] that in the case where τ starts with 1 and $\mathrm{des}(\tau) = 1$, every fixed point O of I_τ has the additional property that the first elements in the bricks of O form an increasing sequence, reading from left to right. Thus we have the following lemma.

Lemma 3.1 *Suppose that $\tau \in S_j$, τ starts with 1, and $\mathrm{des}(\tau) = 1$. Let $\theta_\tau : \Lambda \to \mathbb{Q}(y)$ be the ring homomorphism defined on Λ where $\mathbb{Q}(y)$ is the set of rational functions in the variable y over the rationals $\mathbb{Q}$, $\theta_\tau(e_0) = 1$, and $\theta_\tau(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1, y)$ for $n \geq 1$. Then*

$$n!\theta_\tau(h_n) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} \mathrm{sgn}(O)W(O) \tag{18}$$

where $\mathcal{O}_{\tau,n}$ is the set of objects and I_τ is the involution defined above. Moreover, $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \ldots, \sigma_n$ is a fixed point of I_τ if and only if O satisfies the following three properties:

1. *there are no cells labeled with y in O, i.e., the elements in each brick of O are increasing,*
2. *the first elements in each brick of O form an increasing sequence, reading from left to right, and*
3. *if b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is increase between b_i and b_{i+1}, i.e., $\sigma_{\sum_{j=1}^i |b_j|} < \sigma_{1+\sum_{j=1}^i |b_j|}$, or (b) there is a decrease between b_i and b_{i+1}, i.e., $\sigma_{\sum_{j=1}^i |b_j|} > \sigma_{1+\sum_{j=1}^i |b_j|}$, but there is τ-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $\sigma_{\sum_{j=1}^i |b_j|}$ and $\sigma_{1+\sum_{j=1}^i |b_j|}$.*

3.2 Proof of Theorem 1.1

Let $\tau = 1p23\dots(p-1)$ where $p \geq 4$. Then by (17), we must show that the coefficients

$$U_{\tau,n}(y) = \sum_{O \in \mathcal{O}_{\tau,n}, I_\tau(O)=O} \mathrm{sgn}(O)W(O)$$

have the following properties:

1. $U_{\tau,1}(y) = -y$, and
2. for $n > 1$,
$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor+1} (-y)^{k-1}\binom{n-(k-1)(p-3)-2}{k-1} U_{\tau,n-(((k-1)(p-2)+1)}(y)$.

Property (1) is immediate since there is only one filled-labeled-brick tabloid O of size 1, namely, $O = ((1),1)$, and $\mathrm{sgn}(O) = -1$ and $W(O) = y$.

Next assume that $n > 1$ and $O = (B,\sigma) \in \mathcal{O}_{\tau,n}$ is a fixed point of I_τ where $B = (b_1,\dots,b_k)$ and $\sigma = \sigma_1\dots\sigma_n$. By Lemma 3.1, we know that 1 is in the first cell of O so that $\sigma_1 = 1$.

We claim that 2 must be in the second or third cell of O. That is, it must be the case that either $\sigma_2 = 2$ or $\sigma_3 = 2$. To prove this, suppose for a contradiction that $\sigma_c = 2$ where $c > 3$. Since there are no descents within any brick, 2 must be in the first cell of its brick. Moreover, since the minimal elements in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick b_2 so that $c = |b_1|+1$. Since $c > 3$, $|b_1| \geq 3$ which implies that $1 < \sigma_{c-2} < \sigma_{c-1}$ and $\sigma_{c-2} > \sigma_c = 2$. However, by part (b) of part 3 of Lemma 3.1, this means that there must be a τ-match that involves σ_{c-1} and σ_c. Since τ has only one descent, this would mean that $\sigma_{c-2}\sigma_{c-1}\sigma_c$ would have to play the role of $1p2$ in the τ-match which is impossible since $\sigma_{c-2} > \sigma_c$.

We now have two cases depending on whether $\sigma_2 = 2$ or $\sigma_3 = 2$.

Case 1. $\sigma_2 = 2$.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of O or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O. In either case, we know that 1 is not part of a $1p23\dots(p-1)$-match in O since 2 cannot play the role of p in $1p23\dots(p-1)$-match in O. It follows that $\mathrm{red}_1(O)$ satisfies conditions (1), (2), and (3) of Lemma 3.1 and, hence, $\mathrm{red}_1(O)$ is a fixed point of I_τ. In case (i), we see that $\mathrm{sgn}(O) = \mathrm{sgn}(\mathrm{red}_1(O))$ and $W(O) = W(\mathrm{red}_1(O))$ and, in case (ii), $\mathrm{sgn}(O) = -\mathrm{sgn}(\mathrm{red}_1(O))$ and $W(O) = yW(\mathrm{red}_1(O))$.

Moreover, we can create a fixed point $O = (B,\sigma) \in \mathcal{O}_n$ satisfying conditions (1), (2) and (3) of Lemma 3.1 where $\sigma_2 = 2$ by starting with a fixed point $(B',\sigma') \in \mathcal{O}_{\tau,n-1}$ of I_τ, where $B' = (b'_1,\dots,b'_r)$ and $\sigma' = \sigma'_1\dots\sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1+1)\dots(\sigma'_{n-1}+1)$, and setting $B = (1,b'_1,\dots,b'_r)$ or setting $B = (1+b'_1,\dots,b'_r)$.

It follows that fixed points in Case 1 will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case 2. $\sigma_3 = 2$.

Since there are no descents within bricks in O and the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of brick b_2. Thus it must be the case that b_1 has two cells and $\sigma_2 > \sigma_3$. By part (b) of condition 3 of Lemma 3.1, there must be exist a τ-match among the elements of bricks b_1 and b_2 that involves σ_2 and σ_3. The only way this is possible is if the τ-match starts in cell 1 so that $\mathrm{red}(\sigma_1 \ldots \sigma_p) = 1p23\ldots(p-1)$. Hence b_2 must have at least $p-2$ cells.

Next we claim that $\sigma_{p-1} = p-2$. That is, we know that σ_{p-1} must be greater than $\sigma_1, \sigma_3, \ldots, \sigma_{p-2}$ so that $\sigma_{p-1} \geq p-2$. Next suppose for a contradiction that $\sigma_{p-1} > p-2$. Then let i be the least number in the set $\{1, \ldots, p-2\}$ that is not contained in bricks b_1 and b_2. Since the numbers in each brick are increasing and the minimal elements of the bricks are increasing, the only possible position for i is the first cell of brick b_3. But then it follows that there is a descent between the last cell of brick b_2 and the first cell of b_3. Since O is a fixed point of I_τ, this must mean that there is a τ-match that includes the last cell of b_2 and the first cell of b_3. But since τ has only one descent, this τ-match can only start at the cell $c = b_1 + b_2 - 1$ which is the penultimate cell of b_2. Thus c could be $p-1$ if b_2 has $p-2$ cells or $c > p-1$ if b_2 has more than $p-2$ cells. In either case, $p-1 \leq \sigma_{p-1} \leq \sigma_c < \sigma_{c+1} > \sigma_{c+2} = i$. But this is impossible since to have a τ-match starting at cell c, we must have $\sigma_c < \sigma_{c+2}$. Thus it must be the case that $\sigma_{p-1} = p-2$ and $\{\sigma_1, \ldots, \sigma_{p-1}\} - \{\sigma_2\} = \{1, \ldots, p-2\}$.

We now have two subcases depending on whether or not there is a τ-match in O starting at cell $p-1$.

Subcase 2.1. There is no τ-match in O starting at cell $p-1$.

First, we claim that in this case $\sigma_p = p-1$. That is, if $\sigma_p \neq p-1$, then $\sigma_p > p-1$. This means that $p-1$ cannot be in brick b_2. Similarly, $p-1$ cannot be σ_2 since the fact that there is a $1p2\ldots(p-1)$-match starting at cell 1 means that $\sigma_2 > \sigma_p > p-1$. Thus $p-1$ must be in the first cell of the brick b_3. This would imply that there is a descent between the last cell of b_2 and the first cell of b_3 since $p-1 < \sigma_p$ and σ_p is in b_2. Since there is no τ-match in O starting at cell $p-1$, the only possible τ-match contained the cells of b_2 and b_3 would have to start at cell c where $c \neq p-1$. It cannot be that $c < p-1$ since then it would be the case that $\sigma_c < \sigma_{c+1} < \sigma_{c+2}$. Also, it cannot be that $c > p-1$ since then $\sigma_c > p-1$ and σ_c must be the least integer in the τ-match. Thus it must be the case that $\sigma_p = p-1$.

Then we have that $O' = \mathrm{red}_{p-1}(O)$ satisfies conditions (1), (2), and (3) of Lemma 3.1. Hence it follows that $O' = \mathrm{red}_{p-1}(O)$ is a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$ such that $\mathrm{sgn}(O) = -\mathrm{sgn}(\mathrm{red}_{p-1}(O))$ and $W(O) = yW(\mathrm{red}_{p-1}(O))$. Note that if b_2 has $p-2$ cells, then O' will start with a brick of size one and if b_2 has more than $p-2$ cells, then O' will start with a brick of size at least two. On the other hand, if we start with fixed point $O' = (B', \sigma') \in \mathcal{O}_{\tau,n-(p-1)}$ of I_τ, then we can construct a filled-brick tabloid $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ satisfying conditions (1), (2), and (3) of Lemma 3.1 which has a τ-match starting at cell 1, but has no τ-match starting at cell $p-1$ in O, by first picking $\sigma_2 \in \{p, \ldots, n\}$, then letting $\sigma_1 = 1$, $\sigma_3 = 2, \sigma_4 = 3, \ldots, \sigma_{p-1} = p-1$ and letting $\sigma_p \cdots \sigma_n$ be permutation of $\{1, \ldots, n\} - \{\sigma_1, \ldots, \sigma_{p-1}\}$ such that $\mathrm{red}(\sigma_p \cdots \sigma_n) = \sigma'$. If $B' = (b_1', b_2', \ldots, b_s')$, we let $B = (2, p-3+b_1', b_2', \ldots, b_s')$. Hence (B, σ) is a fixed point of I_τ. It follows that the fixed points in Subcase 2.1 will contribute $(-y)(n-(p-1))U_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Subcase 2.2. There is a τ-match starting at cell $(p-1)$ in O.

In this subcase, it must be that $\sigma_{p-1} < \sigma_p > \sigma_{p+1}$ so that b_2 must have $p-2$ cells and brick b_3 starts at cell $p+1$. We claim that b_3 must have at least $p-2$ cells. That is, there must be a τ-match that includes the last cell of b_2 and the first cell of b_3 and since the pattern τ is of length p, b_3 must have at least $p-2$ cells. Moreover, there is a τ-match that starts at cell 1 and another one that starts at cell $p-1$. These two τ-matches overlap on σ_{p-1} and σ_p. In general, assume that there is chain of τ-matches in $O=(B,\sigma)$ starting at cell 1 where each consecutive pair of τ-matches overlap on two cells. Suppose there are exactly $k-1$ such τ-matches in this chain. Then the jth τ-match starts at the penultimate cell of brick b_j. Brick b_1 must have two cells and brick b_j must have $p-2$ cells for each $2 \leq j \leq k-1$ and brick b_k must have at least $p-2$ cells. Let r_j be an integer such that the jth τ-match starts at cell r_j. Thus $r_j = 1+(j-1)(p-2)$ for $1 \leq j \leq k-1$. Define $r_k = 1+(k-1)(p-2)$ and assume that O does not have a τ-match starting at position r_k. Thus we have the situation pictured below in Figure 3.2.

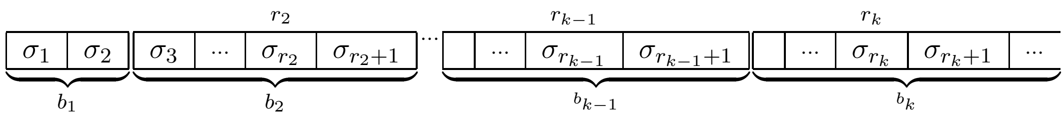

Figure 6: An example of a brick tabloid with a chain of $k-1$ τ-matches each starting at r_j.

First we claim $\sigma_{r_j} = r_j - (j-1)$ and

$$\{1,\ldots,r_j-(j-1)\} = \{\sigma_1,\ldots,\sigma_{r_j}\} - \{\sigma_{r_i+1} : i=1,\ldots,j-1\}$$

for $j=1,\ldots,k$. We have shown that $\sigma_1 = 1$ and that $\sigma_{r_2} = \sigma_{p-1} = p-2$ and $\{\sigma_1,\ldots,\sigma_{p-1}\} - \{\sigma_2\} = \{1,\ldots,p-2\}$. Thus assume by induction, $\sigma_{r_{j-1}} = r_{j-1}-(j-2)$ and $\{1,\ldots,r_{j-1}-(j-2)\} = \{\sigma_1,\ldots,\sigma_{r_{j-1}}\} - \{\sigma_{r_i+1} : i=1,\ldots,j-2\}$. Since there is a τ-match that starts at cell r_{j-1} and $p \geq 4$, we know that all the integers in

$$\{\sigma_{r_{j-1}},\sigma_{r_{j-1}+1},\sigma_{r_{j-1}+2},\ldots,\sigma_{r_{j-1}+p-3}\} - \{\sigma_{r_{j-1}+1}\}$$

are less than $\sigma_{r_j} = \sigma_{r_{j-1}+p-2}$. Since

$$\{1,\ldots,r_{j-1}-(j-2)\} = \{\sigma_1,\ldots,\sigma_{r_{j-1}}\} - \{\sigma_{r_i+1} : i=1,\ldots,j-2\},$$

it follows that $\sigma_{r_j} \geq r_{j-1}-(j-2)+(p-3) = r_j-(j-1)$.

Next suppose that $\sigma_{r_j} > r_j-(j-1)$. Then let i be the least integer which is in

$$\{1,\ldots,r_j-(j-1)\} - (\{\sigma_1,\ldots,\sigma_{r_j}\} - \{\sigma_{r_i+1} : i=1,\ldots,j-1\}).$$

Our assumptions ensure that $\sigma_{r_1+1} > \sigma_{r_2+1} > \ldots > \sigma_{r_j+1}$ so that i does not lie in the bricks $b_1,\ldots,b_j$. Because the integers in each brick increase and the minimal integers in the bricks are increasing, it must be the case that i is in the first cell of the next brick b_{j+1}. Now it cannot be that $j<k$ because then we have that $i = \sigma_{r_j+2} \leq r_j-(j-1) < \sigma_{r_j} < \sigma_{r_j+1}$ which would violate the fact that there is a τ-match in O starting at cell r_j. If $j=k$, then it follows that there is a descent between the last cell of b_k and the first cell of b_{k+1} since i is in the first cell of b_{k+1} and $i \leq r_k-(k-1) < \sigma_{r_k}$. Since O is a fixed point of I_τ, this must mean that there is a τ-match that includes the last cell of b_k and the first cell of b_{k+1}. But since τ has only one descent, this τ-match can only start at the cell c which is the penultimate cell of b_k. Thus c must be greater than r_k because we are assuming that there is no τ-match starting at cell r_k. Hence b_{k+1} must have more than $p-2$ cells. In this case, we have that $i \leq r_k - (k-1) < \sigma_{r_k} \leq \sigma_c < \sigma_{c+1} > \sigma_{c+2} = i$. But this cannot be since to have a τ-match starting at cell c, we must have $\sigma_c < \sigma_{c+2}$. Therefore it is not true that $\sigma_{r_j} > r_j - (j-1)$ so that it must be the case that $\sigma_{r_j} = r_j - (j-1)$. Finally since

1. $\{1, \ldots, r_{j-1} - (j-2)\} = \{\sigma_1, \ldots, \sigma_{r_{j-1}}\} - \{\sigma_{r_i+1} : i = 1, \ldots, j-2\}$ and
2. $\sigma_{r_{j-1}}, \sigma_{r_{j-1}+2}, \ldots, \sigma_{r_{j-1}+p-3} < \sigma_{r_j}$,

it must be the case that

$$\{1, \ldots, r_j - (j-1)\} = \{\sigma_1, \ldots, \sigma_{r_j}\} - \{\sigma_{r_i+1} : i = 1, \ldots, j-1\}$$

as desired. Thus we have proved by induction that $\sigma_{r_j} = r_j - (j-1)$ and $\{1, \ldots, r_j - (j-1)\} = \{\sigma_1, \ldots, \sigma_{r_j}\} - \{\sigma_{r_i+1} : i = 1, \ldots, j-1\}$ for $j = 1, \ldots, k$.

This means that the positions of the elements in the set $\{\sigma_1, \ldots, \sigma_{r_k}\} - \{\sigma_{r_i+1} : i = 1, \ldots, k-1\}$ are completely determined. Next we claim that since there is no τ-match starting at position r_k, it must be the case that $\sigma_{r_k+1} = r_k - (k-1) + 1 = r_k - k + 2$. That is, since there is a τ-match starting at cell r_j for $j = 1, \ldots, k-1$, it must be the case that $\sigma_{r_1+1} > \sigma_{r_2+1} > \cdots > \sigma_{r_{k-1}+1} > \sigma_{r_k+1}$. If $\sigma_{r_k+1} \neq r_k - k + 2$, then $\sigma_{r_k+1} > r_k - k + 2$ and, hence, $r_k - k + 2$ cannot be in any of the bricks $b_1, \ldots, b_k$. Thus $r_k - k + 2$ must be in the first cell of the brick b_{k+1}. But then there will be a descent between the last cell of b_k and the first cell of b_{k+1} since $r_k - k + 2 < \sigma_{r_k+1}$ and σ_{r_k+1} is in b_k. Since there is no τ-match starting at cell r_k, the only possible τ-match among the cells of b_k and b_{k+1} would have to start at a cell c with $c \neq r_k$. But it cannot be that $c < r_k$ since then $\sigma_c < \sigma_{c+1} < \sigma_{c+2}$. Similarly, it cannot be that $c > r_k$ since then $\sigma_c > r_k - k + 2$ and $r_k - k + 2$ would have to be part of the τ-match which means that σ_c could not play the role of 1 in the τ-match. Thus it must be the case that $\sigma_{r_k+1} = r_k - k + 2$.

It follows that $O' = \text{red}_{r_k}(O)$ satisfies conditions (1), (2), and (3) of Lemma 3.1 and hence is a fixed point of I_τ in $\mathcal{O}_{\tau,n-r_k}$. Note that if b_k has $p-2$ cells, then the first brick of O' will be of size 1 and if b_k has more than $p-2$ cells, then the first brick of O' will have size at least two. Since there is a τ-match starting at each of the cells r_j for $j = 1, \ldots, k-1$, it must be the case that $\sigma_{r_1+1} > \sigma_{r_2+1} > \cdots > \sigma_{r_{k-1}+1} > \sigma_{r_k+1} = r_k - k + 2$. On the other hand, if we start with any fixed point $O' = (B', \sigma') \in \mathcal{O}_{\tau,n-r_k}$ of I_τ where $B' = (b'_1, \ldots, b'_s)$, then we can create filled-labeled-brick tabloid $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ satisfying conditions (1), (2), and (3) of Lemma 3.1 which has τ-matches starting at positions $1, r_1, \ldots, r_{k-1}$ but no τ-match starting at position r_k by letting the first $k-1$ bricks of B be a brick of size 2 followed by $k-2$ bricks of size $p-2$ and then having the k-th brick of B be a size $p-2+b'_1$ and the remaining bricks be $b'_2, \ldots, b'_s$. The permutation σ is constructed by ensuring that

(i) the elements $1, \ldots, r_k - k - 2$ occupy the set of cells $\{1, \ldots, r_k + 1\} - \{r_i + 1 : i = 1, \ldots, k-1\}$, reading from left to right,

(ii) there are $k-1$ integers $a_1 > \cdots > a_{k-1}$ from $\{r_k + k - 1, \ldots, n\}$ which occupy cells $r_1 + 1, \ldots, r_{k-1} + 1$, reading from left to right, and

(iii) $\sigma_{r_k-k+1}\ldots\sigma_n$ is a permutation of $\{1,\ldots,n\}-(\{1,\ldots,r_k-k+2\}\cup\{a_1,\ldots,a_{k-1}\})$ that reduces to σ'.

Each such (B,σ) will be a fixed point of I_τ such that σ has a chain of τ-matches that start at σ_1 and overlap in exactly two elements. Note that we have $\binom{n-(r_k-k-2)}{k-1}=\binom{n-(k-1)(p-3)-2}{k-1}$ ways choose the numbers $a_1,\ldots,a_{k-1}$. Moreover, $W(O)=y^{k-1}W(O')$ and $\mathrm{sgn}(O)=(-1)^{k-1}\mathrm{sgn}(O')$. It follows that the fixed points in Subcase 2.2 will contribute

$$\sum_{k\geq 3}^{\lfloor\frac{n-2}{p-2}\rfloor+1}(-y)^{k-1}\binom{n-(k-1)(p-3)-2}{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y)$$

to $U_{\tau,n}(y)$.

Hence we have proved that if $\tau=1p23\ldots(p-1)$ where $p\geq 4$, then

$$NM_\tau(t,x,y)=\left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y)=1+\sum_{n\geq 1}U_{\tau,n}(y)\frac{t^n}{n!},$$

$U_{\tau,1}(y)=-y$, and, for $n>1$,

$$U_{\tau,n}(y)=(1-y)U_{\tau,n-1}(y)+\sum_{k=2}^{\lfloor\frac{n-2}{p-2}\rfloor+1}(-y)^{k-1}\binom{n-(k-1)(p-3)-2}{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y).$$

Thus Theorem 1.1 holds as desired.

For any polynomial $f(x)$, we let $f(x)|_{x^k}$ denote the coefficient of x^k in $f(x)$. We have used Theorem 1.1 to compute tables of the coefficients $U_{1p23\ldots p,n}(y)|_{y^i}$ for $n\leq 8$ and $p=4,5,6$. Having determined these polynomials $U_{1p23\ldots p,n}(y)$, we have used Mathematica to compute tables of the polynomials $NM_{1p23\ldots p-1,n}(x,y)$ for $n\leq 8$ and $p=4,5,6$. See Tables 2-7 in Appendix I.

We note that there are many terms in these expansions which are easily explained. For example, the identity permutation $\epsilon=12\ldots n$ corresponds to the term xy in $NM_{1p23\ldots(p-1),n}(x,y)$ and the reverse of the identity permutations $\overline{\epsilon}=n(n-1)\ldots 21$ corresponds to the term x^ny^n in $NM_{1p23\ldots(p-1),n}(x,y)$. More generally, we claim that $NM_{1p2\ldots(p-1),n}(x,y)|_{x^ky^k}$ is always the Stirling number of the second kind $S(n,k)$ which is the number of set partitions of $\{1,\ldots,n\}$ into k parts. That is, a permutation $\sigma\in S_n$ that contributes to the coefficient x^ky^k in $NM_{1p2\ldots(p-1),n}(x,y)$ must have k left-to-right minima and $k-1$ descents. Since each left-to-right minima of σ which is not the first element is always the second element of a descent pair, it follows that if $1=i_1<i_2<i_3<\cdots<i_k$ are the positions of the left to right minima, then σ must be increasing in each of the intervals $[1,i_2),[i_2,i_3),\ldots,[i_{k-1},i_k),[i_k,n]$. But this means that

$$\{\sigma_1,\ldots,\sigma_{i_2-1}\},\{\sigma_{i_2},\ldots,\sigma_{i_3-1}\},\ldots,\{\sigma_{i_{k-1}},\ldots,\sigma_{i_k-1}\},\{\sigma_{i_k},\ldots,\sigma_n\}$$

is just a set partition of $\{1,\ldots,n\}$ ordered by decreasing minimal elements. Moreover, no such permutation can have a $1p2\ldots(p-1)$-match for any $p\geq 4$. Vice versa, if $A_1,\ldots,A_k$ is a set partition of $\{1,\ldots,n\}$ such that $min(A_1)>\cdots>min(A_k)$, then the permutation $\sigma=A_k\uparrow A_{k-1}\uparrow\ldots A_1\uparrow$ is a permutation with k left-to-right minima and $k-1$ descents where for any set $A\subseteq\{1,\ldots,n\}$, $A\uparrow$ is the list of the elements of A in increasing order. It follows that for any $p\geq 4$,

1. $NM_{1p2\ldots(p-1),n}(x,y)|_{xy} = S(n,1) = 1$,
2. $NM_{1p2\ldots(p-1)}(x,y)|_{x^2y^2} = S(n,2) = 2^{n-1} - 1$,
3. $NM_{1p2\ldots(p-1)}(x,y)|_{x^ny^n} = S(n,n) = 1$, and
4. $NM_{1p2\ldots(p-1)}(x,y)|_{x^{n-1}y^{n-1}} = S(n,n-1) = \binom{n}{2}$.

It is also not difficult to determine $NM_{1p23\ldots(p-1),n}(x,y)|_{xy^2}$ which corresponds to those permutations $\sigma \in \mathcal{NM}_{1p23\ldots(p-1),n}$ which have one descent and $\text{LRmin}(\sigma) = 1$ so that the first element of σ must be 1. First we can create a permutation with one descent by picking any subset $A \subseteq \{1,\ldots,n\}$ and letting $\sigma_{A,n}$ be the permutation which consists of the elements of A in increasing order followed by the elements of $\{1,\ldots,n\} - A$ in increasing order. For example if $n = 6$ and $A = \{2,4\}$, then $\sigma_{A,6} = 241356$. Clearly if A equals $\emptyset$ or $A = \{1,\ldots,i\}$ for some $i \leq n$, then $\sigma_{A,n}$ is just the identity permutation which has no descents. Thus the number of permutations of S_n with exactly one descent is $2^n - n - 1$. We have already shown that the number of permutations $\sigma \in S_n$ such that $\text{des}(\sigma) = 1$ and σ does not start with 1 is $2^{n-1} - 1$. It follows that there are $2^n - (n+1) - (2^{n-1} - 1) = 2^{n-1} - n$ permutations $\sigma \in S_n$ that have one descent and start with 1. Next we have to determine the number of permutations of S_n that have one descent and start with 1 which have a $1p23\ldots(p-1)$-match. Clearly if $n < p$, then there are no $\sigma \in S_n$ such that $\text{des}(\sigma) = 1$ and σ has a $1p23\ldots(p-1)$-match. If $1 \in A$ and $\sigma_{A,n}$ does have $1p23\ldots(p-1)$-match, then the largest 2 elements of A must play the role of 1 and p in the $1p23\ldots(p-1)$-match in σ_A and first $(p-2)$ elements of $\{1,\ldots,n\} - A$ must play the role of $2\ldots(p-1)$ in the $1p23\ldots(p-1)$-match in σ_A. It follows that all but the largest element of A must be smaller than all the elements $\{1,\ldots,n\} - A$ and the first $(p-2)$ elements of $\{1,\ldots,n\} - A$ must be smaller than the largest element of A. Hence $\sigma_{A,n}$ must be of the form $1\ldots s\ (s+1)\ x\ (s+2)\ (s+3)\ldots(s+p-1)\ldots$ where $x > (s+p-1)$ for some $0 \leq s \leq n-p$. Thus for any given s, we have $n-(s+p-1)$ choices for x. Hence if $n \geq p$, the number of $\sigma \in S_n$ which have one descent, start with 1, and have one $1p23\ldots(p-1)$-match is

$$\sum_{s=0}^{n-p} (n-(s+p-1)) = \binom{n-p+2}{2}.$$

It then follows that for all $p \geq 4$,

$$NM_{1p23\ldots p-1,n}(x,y)|_{xy^2} = \begin{cases} 2^{n-1} - n & \text{for } n < p \\ 2^{n-1} - n - \binom{n-p+2}{2} & \text{for } n \geq p. \end{cases}$$

3.3 Proof of Theorem 1.2

Let $\tau = 13\ldots(p-1)2p$ where $p \geq 4$. Then we want to show that $U_{\tau,1}(y) = -y$ and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor} (-y)^k \frac{1}{(p-2)k+1}\binom{k(p-1)}{k} U_{\tau,n-(k(p-2)+1)}(y). \quad (19)$$

Again we must study the fixed points of I_τ for $\tau = 13\ldots(p-1)2p$ where $p \geq 4$.

Let $O = (B, \sigma)$ be a fixed point of I_τ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. By Lemma 3.1, we know that $\sigma_1 = 1$. We claim that $\sigma_2 = 2$ or $\sigma_{p-1} = 2$. To show this, suppose that $\sigma_c = 2$ where $c \notin \{2, p-1\}$. Since there are no descents within any brick, 2 must be in the first cell of a brick. Moreover, since the minimal elements in the bricks of O form an increasing sequence, reading from left to right, 2 must be in the first cell of b_2. Thus 1 is in the first cell of b_1 and 2 is in the first cell of b_2. Then there is a descent between the last cell of b_1 and the first cell of b_2. By Lemma 3.1, this means that there is a τ-match in σ contained in the cells in bricks b_1 and b_2 that involves the element in the last cell of b_1, namely σ_{c-1}, and the element in the first cell of b_2, namely $\sigma_c = 2$. Clearly, σ_c must play the role of 2 in the τ-match which means that the τ-match starts at cell 1 since only 1 can play the role of the 1 in the τ-match. But then it follows that c must be equal to $(p-1)$ which contradicts our choice of c. Hence it must be the case that $\sigma_c = 2$ where $c \in \{2, p-1\}$. Thus we have two cases to consider.

Case I. $\sigma_2 = 2$.

In this case, we can use the same argument that we used in Case 1 above to show that the fixed points in Case I will contribute $(1-y)U_{\tau,n-1}(y)$ to $U_{\tau,n}(y)$.

Case II. $\sigma_{p-1} = 2$.

Then σ_{p-1} must be the first cell of b_2 so that b_1 has $p-2$ cells and $\sigma_{p-2} > \sigma_{p-1} = 2$. By condition 3 of Lemma 3.1, there must be a τ-match that involves σ_{p-2} and σ_{p-1} contained in the cells of b_1 and b_2 which means that b_2 must contain at least 2 cells. We now have two subcases based on whether or not there is a τ-match in O starting at cell $(p-1)$.

Subcase II.1. There is no τ-match in O starting at cell $(p-1)$.

We claim that $\sigma_p = p$. First observe that σ_p must be greater than or equal to $\sigma_1, \dots, \sigma_{p-1}$ since there is a τ-match starting at cell 1. Thus $\sigma_p \geq p$. If $\sigma_p > p$, then p cannot be in brick b_2. Since brick b_1 has $p-2$ cells and 1 is in b_1, we cannot have all of the integers $3, \dots, p$ in b_1 so let i be the least integer in $\{3, \dots, p\}$ which is not in b_1. Since $\sigma_p > p$, we know i cannot be in brick b_2. Since the minimal elements in the bricks are increasing, it must be the case that i is in the first cell of brick b_3 and there is a descent between the last cell of b_2 and the first cell of b_3. This implies that there is a τ-match that includes the last cell of b_2 and the first cell of b_3. Since we are assuming there is no τ-match starting at cell $p-1$, this τ-match must start at some cell c where $c > p-1$. But this is impossible since i which is in the first cell of b_3 must play the role of 2 in that τ-match and σ_c must play the role of 1 in that τ-match and we know that $i \leq p < \sigma_p \leq \sigma_c$. Hence it must be the case that $\sigma_p = p$ which forces that $\sigma_1 \dots \sigma_p = 13 \dots (p-1)2p$.

Then it is not hard to check that $\text{red}_{p-1}(O)$ satisfies conditions (1), (2), and (3) of Lemma 3.1 and hence it is a fixed point of I_τ in $\mathcal{O}_{\tau,n-(p-1)}$ such that $W(O) = yW(O')$ and $\text{sgn}(O) = -\text{sgn}(O')$. Vice versa, if we are given a fixed point of $O' = (B', \sigma') \in \mathcal{O}_{\tau,n-(p-1)}$ of I_τ satisfying conditions (1), (2), and (3) of Lemma 3.1 where $B' = (b'_1, \dots, b'_s)$ and $\sigma' = \sigma'_1 \dots \sigma'_{n-(p-1)}$, then we can construct a fixed point $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ of I_τ such that O has a τ-match starting at cell 1 but does not have a τ-match starting at cell $(p-1)$, be letting $B = (p-2, 2+b'_1, \ldots, b'_s)$ and $\sigma = 13\ldots(p-1)2p(\sigma'_2+p)\ldots(\sigma'_{n-p-1}+p)$. It follows that fixed points in Subcase II.1 will contribute $(-y)U_{\tau,n-(p-1)}(y)$ to $U_{\tau,n}(y)$.

Subcase II.2. There is a τ-match starting at cell $(p-1)$.

In this case, it must be that $\sigma_{p-1} < \sigma_{2p-3} < \sigma_p < \cdots < \sigma_{2p-4}$. We claim that b_2 must be of size $p-2$. Clearly, b_2 has at most $p-2$ cells since the elements in each brick are increasing and $\sigma_{2p-4} > \sigma_{2p-3}$. Now if b_2 has less than $p-2$ cells, then cell $(2p-3)$ must be the first cell of some brick b_k and brick b_{k-1} would have less than $p-2$ cells. Then all the elements in b_{k-1} are strictly larger than the first element of b_k so that it would not be possible to have a τ-match contained in the bricks b_{k-1} and b_k which would contradict the fact that O is a fixed point of I_τ by Lemma 3.1. Thus brick b_2 has $p-2$ cells which, in turn, implies that brick b_3 must have at least 2 cells. That is, if b_3 has less than 2 cells, there could be no τ-match among the cells of b_2 and b_3 even though there is a descent between the last cell of b_2 and the first cell of b_3 violating the fact that O is a fixed point of I_τ.

Notice here that there is a τ-match that starts at cell 1 and another one that starts at cell $(p-1)$. These two τ-matches overlap on the cells $(p-1)$ and p. In general, assume that there is a chain of τ-matches starting at cell 1 that each overlap by two cells. Suppose there are exactly $k-1$ such τ-matches in this chain. Then the jth τ-match starts at the first cell of brick b_j. Brick b_j must have $p-2$ cells for each $1 \leq j \leq k-1$ and brick b_k must have at least 2 cells. Let r_j be an integer such that the jth τ-match starts at cell r_j. Then it follows that $r_j = 1+(j-1)(p-2)$ for $1 \leq j \leq k-1$. Define $r_k = 1+(k-1)(p-2)$ and assume that O does not have a τ-match starting at cell r_k.

Next we claim that $\sigma_{r_k+1} = r_k+1$ and $\{\sigma_1, \ldots, \sigma_{r_k}, \sigma_{r_k+1}\} = \{1, \ldots, r_k+1\}$. That is, since there are τ-matches starting at positions $r_1, r_2, \ldots, r_{k-1}$, we have that $\sigma_{r_j}, \ldots, \sigma_{r_{j+1}} < \sigma_{r_{j+1}+1}$ for each $1 \leq j \leq k-1$. It follows that σ_{r_k+1} is greater than σ_i for $i = 1, \ldots, r_k$ so that $\sigma_{r_k+1} \geq r_k+1$. For a contradiction, assume that $\sigma_{r_k+1} > r_k+1$. It then follows that there is at least one $i \in \{1, \ldots, r_k+1\}$ which does not appear in the first r_k+1 cells of O so let j be the least element in $\{1, \ldots, r_k+1\} - \{\sigma_1, \ldots, \sigma_{r_k+1}\}$. Then j cannot lie in brick b_k because $j < \sigma_{r_k+1}$ and brick b_k starts at cell r_k+1 so that j must be in the first cell of brick b_{k+1}. Thus there is a descent between the last cell of b_k and the first cell of b_{k+1}. But then we claim that there can be no τ-match that includes the last cell of b_k the first cell of b_{k+1}. That is, we are assuming that there is no τ-match starting at cell r_k in O. Thus if there is a τ-match contained in the cells of b_k and b_{k+1}, it must start after position r_k and involve j. But j is less than all the integers in brick b_k that appear after cell r_k which means that none of them can play the role of 1 in such a τ-match. This violates the fact that O is a fixed point of I_τ. Thus it must be the case that $\sigma_{r_k+1} = r_k+1$. Since σ_{r_k+1} is greater than σ_i for $i = 1, \ldots, r_k$, it automatically follows that $\{\sigma_1, \ldots, \sigma_{r_k+1}\} = \{1, \ldots, r_k+1\}$.

It is then easy to check that $O' = \text{red}_{r_k}(O)$ satisfies conditions (1), (2), and (3) of Lemma 3.1 and hence it is a fixed point of I_τ in $\mathcal{O}_{\tau,n-r_k}$. Moreover, since each of the first $k-1$ bricks contributes a factor of $-y$ to $\text{sgn}(O)W(O)$, we have that $\text{sgn}(O')W(O') = (-y)^{k-1}\text{sgn}(O)W(O)$. On the other hand, if we start with fixed point $O' = (B', \sigma') \in \mathcal{O}_{\tau,n-r_k}$ of I_τ where $B' = (b'_1, \ldots, b'_s)$ and $\sigma' = \sigma'_1 \ldots \sigma'_{n-r_k}$, then we can create an $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ satisfying conditions (1), (2), and (3) of Lemma 3.1 which necessarily will be a fixed point of I_τ where O has τ-matches starting at positions $1, r_2, \ldots, r_{k-1}$ but not at r_k by letting $\sigma = \sigma_1 \ldots \sigma_{r_k+1}(r_k+\sigma'_2)\ldots(r_k+\sigma'_{n-r_k})$ where $\sigma_1 \ldots \sigma_{r_k+1}$ is a permutation of S_{r_k+1} which has τ-matches starting at positions $1, r_2, \ldots, r_{k-1}$ and letting B start out with $k-1$ bricks of size $p-2$ followed by a brick of size $p-2+b'_1$ followed by $b'_2, \ldots, b'_s$. It follows that the contribution of such fixed points to $U_{\tau,n}(y)$ is $(-y)^{k-1}D_{\tau,r_k+1}U_{\tau,n-(k-1)(p-2)+1)}(y)$ where D_{τ,r_k+1} is the number of $\sigma \in S_{r_k+1}$ such that there are τ-matches in σ starting at positions $1, r_2, \ldots, r_{k-1}$.

Fortunately, Harmse proved a formula in his thesis [10] from which we can obtain a formula for D_{τ,r_k+1} for any $\tau = 13\ldots(p-1)2p$ where $p \geq 4$. In particular, this formula was needed for the study of column strict fillings of rectangular shapes initiated by Harmse and the second author [11]. That is, Harmse and the second author [11] defined $\mathcal{F}_{n,k}$ to be the set of all fillings of a $k \times n$ rectangular array with the integers $1, \ldots, kn$ such that that the elements increase from bottom to top in each column. We let (i,j) denote the cell in the i^{th} row from the bottom and the j^{th} column from the left of the $k \times n$ rectangle and we let $F(i,j)$ denote the element in cell (i,j) of $F \in \mathcal{F}_{n,k}$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $0 < \lambda_1 \leq \cdots \leq \lambda_k$, we let F_λ denote the Ferrers diagram of λ, i.e. F_λ is the set of left-justified rows of squares where the size of the i-th row is λ_i. Thus a $k \times n$ rectangular array corresponds to the Ferrers diagram corresponding to (n^k). If $F \in \mathcal{F}_{n,k}$ and the integers are increasing in each row, reading from left to right, then F is a standard tableau of shape (n^k). We let St_{n^k} denote the set of all standard tableaux of shape (n^k) and let $st_{n^k} = |St_{n^k}|$. One can use the Frame-Robinson-Thrall hook formula [8] to show that

$$st_{n^k} = \frac{(kn)!}{\prod_{i=0}^{k-1}(i+n)\downarrow_n} \tag{20}$$

where $(n)\downarrow_0 = 1$ and $(n)\downarrow_k = n(n-1)\cdots(n-k+1)$ for $k > 0$.

If F is any filling of a $(k \times n)$-rectangle with distinct positive integers such that elements in each column increase, reading from bottom to top, then we let $\text{red}(F)$ denote the element of $\mathcal{F}_{n,k}$ which results from F by replacing the i^{th} smallest element of F by i. For example, Figure 7 demonstrates a filling, F, with its corresponding reduced filling, $\text{red}(F)$.

F

12	16	22
8	15	17
6	10	13
1	7	5

$\text{red}(F)$

7	10	12
5	9	11
3	6	8
1	4	2

Figure 7: An example of $F \in \mathcal{F}_{3,4}$ and $\text{red}(F)$.

If $F \in \mathcal{F}_{n,k}$ and $1 \leq c_1 < \cdots < c_j \leq n$, then we let $F[c_1, \ldots, c_j]$ be the filling of the $(k \times j)$-rectangle where the elements in column a of $F[c_1, \ldots, c_j]$ equal the elements in column c_a in F for $a = 1, \ldots, j$. Let P be an element of $\mathcal{F}_{j,k}$ and $F \in \mathcal{F}_{n,k}$ where $j \leq n$. Then we say there is a *P-match in F starting at position i* if $\text{red}(F[i, i+1, \ldots, i+j-1]) = P$. We let $P\text{-mch}(F)$ denote the number of P-matches in F.

If $P \in \mathcal{F}_{2,s}$, then we define $\mathcal{MP}_{P,n}$ to be the set of $F \in \mathcal{F}_{n,s}$ such that $P\text{-mch}(F) = n-1$, i.e. the set of $F \in \mathcal{F}_{n,s}$ such that there is a P-match in F starting at positions $1, 2, \ldots, n-1$. Elements of $\mathcal{MP}_{P,n}$ are called maximum packings for P. We let $mp_{P,n} = |\mathcal{MP}_{P,n}|$ and use the convention that $mp_{P,1} = 1$. For example, if P is the element of $\mathcal{F}_{2,k}$ that has the integers $1, \ldots, s$ in the first column and the integers $s+1, \ldots, 2s$ in the second column, then it follows that $mp_{P,n} = 1$ for all $n \geq 1$, since the only element of $F \in \mathcal{F}_{n,k}$ with $P\text{-mch}(F) = n-1$ has the integers $(i-1)s+1, \ldots, (i-1)s+s$ in the i-th column, for $i = 1, \ldots, n$. Harmse and the second author [11] proved that the computation of the generating function for the number of P-matches in $F_{n,k}$ can be reduced to computing $mp_{P,n}$ for all n so that they computed $mp_{P,n}$ for various $P \in \mathcal{F}_{2,k}$. In particular, let $P_s \in St_{2^s}$ be the standard tableau which has $1, 3, 4, \ldots, s+1$ in the first column and $2, s+2, s+3, \ldots, 2s$ in the second column. For example, P_5 is pictured in Figure 8.

6	**10**
5	**9**
4	**8**
3	**7**
1	**2**

Figure 8: The standard tableau P_5.

Then Harmse proved that for $s, n \geq 2$,

$$mp_{P_s,n} = \frac{1}{(s-1)(n-1)+1} \binom{s(n-1)}{n-1} \tag{21}$$

We claim that we can use (21) to obtain our desired formula for D_{τ, r_k+1}. That is, suppose that $s, n \geq 2$ and $F \in \mathcal{MP}_{P_s,n}$. Then in F, the top $s-1$ elements of column i where $i > 1$ are larger than any of the elements in column $i-1$ and are greater than or equal to $F(1,i)$. It follows that the top $s-1$ elements in column n are greater than all the remaining elements in F so that they must be $s(n-1)+2, s(n-1)+3, \ldots, sn$ reading from bottom to top. Given such an F, we let σ_F be the permutation in $S_{s(n-1)+2}$ where

$$\sigma_F = F(1,1)F(2,1)\ldots F(s,1)\ldots F(1,n-1)F(2,n-1)\ldots F(s,n-1)F(1,n)F(2,n).$$

For example, if F is the element of $\mathcal{MP}_{P_5,4}$ pictured at the top of Figure 9, then σ_F is pictured at the bottom of Figure 9.

F=

8	**12**	**16**	**20**
6	**11**	**15**	**19**
5	**10**	**14**	**18**
4	**9**	**13**	**17**
1	**2**	**3**	**7**

σ_F = **1 4 5 6 8 2 9 10 11 12 3 13 14 15 16 7 17**

Figure 9: An example of σ_F.

Then it follows that if $F \in \mathcal{MP}_{P_s,n}$, then σ_F is a permutation in $S_{s(n-1)+2}$ which has $1\ 3 \ldots (s-1)\ 2$ s-matches starting at positions $1+(s-2)(j-1)$ for $j = 1, \ldots, n-1$. Vice versa, if $\sigma \in S_{s(n-1)+2}$ is a permutation which has $1\ 3 \ldots (s-1)\ 2$ s-matches starting at positions $1+(s-2)(j-1)$ for $j = 1, \ldots, n-1$, then we can create a filling of $F_\sigma \in \mathcal{MP}_{P_s,n}$ by letting r^{th} column of F consist of $\sigma_{s(r-1)+1}, \ldots, \sigma_{s(r-1)+s}$, reading from bottom to top, for $r = 1, \ldots, n-1$ and letting the n^{th} column consist of $\sigma_{s(n-1)+1}, \sigma_{s(n-1)+2}, s(n-1)+3, \ldots, sn$, reading from bottom to top. It then follows from

(21) that the number of permutations $\sigma \in S_{(k-1)(p-1)+2}$ that have $1\,3\ldots(p-1)\,2\,p$-matches starting at positions $1+(p-2)(j-1)$ for $j=1,\ldots,k-1$ is $\frac{1}{(p-2)(k-1)+1}\binom{(k-1)(p-1)}{k-1}$. Hence if $\tau = 1\,3\ldots(p-1)\,2\,p$, then

$$D_{\tau,r_k+1} = \frac{1}{(p-2)(k-1)+1}\binom{(k-1)(p-1)}{k-1}.$$

Thus the contribution to $U_{\tau,n}(y)$ of those fixed points O such that the bricks $b_1,\ldots,b_{k-1}$ all have $p-2$ cells and there is a τ-match starting at cell r_j for $1 \leq j \leq k-1$, but there is no τ-match starting at position $r_k = (k-1)(p-2)+1$ is

$$(-y)^{k-1}\frac{1}{(p-2)(k-1)+1}\binom{(k-1)(p-1)}{k-1}U_{\tau,n-((k-1)(p-2)+1)}(y).$$

Hence we have shown that if $\tau = 13\ldots(p-1)2p$ where $p \geq 4$, then $U_{\tau,1}(y) = -y$ and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{p-2} \rfloor}(-y)^k\frac{1}{(p-2)k+1}\binom{k(p-1)}{k}U_{\tau,n-(k(p-2)+1)}(y). \tag{22}$$

This proves Theorem 1.2.

In Tables 8-13 in Appendix 1, we have also computed the values of the polynomials $U_{13\ldots(p-1)2p,n}(y)$ and $NM_{13\ldots(p-1)2p,n}(x,y)$ for $n \leq 8$ and $p = 4,5,6$.

Again, we explain several of these coefficients. For example, the same argument that we used to prove that $NM_{1p2\ldots,(p-1),n}(x,y)|_{x^ky^k} = S(n,k)$ will prove that

$$NM_{13\ldots,(p-1)2p,n}(x,y)|_{x^ky^k} = S(n,k).$$

We claim that for $p \geq 4$,

$$NM_{13\ldots(p-1)2p,n}(x,y)|_{xy^2} = \begin{cases} 2^{n-1}-n & \text{if } n < p \text{ and} \\ 2^{n-1}-2n+p-1 & \text{if } n \geq p. \end{cases} \tag{23}$$

That is, suppose that $\sigma \in S_n$ contributes to $NM_{13\ldots(p-1)2p,n}(x,y)|_{xy^2}$. Then σ must have 1 left-to-right minima and one descent. It follows that σ must start with 1 and have one descent. We have shown that there are $2^{n-1}-n$ permutations that start with 1 and have 1 descent. Next consider when such a σ which starts with 1 and has 1 descent can have a $13\ldots(p-1)2p$-match. If the $13\ldots(p-1)2p$-match starts at position i, then it must be the case that $\sigma_{i+p-3} > \sigma_{i+p-2}$. Thus it follows that $\sigma_1,\ldots,\sigma_{i+p-3}$ and $\sigma_{i+p-2},\ldots,\sigma_n$ are increasing sequences. But the fact that there is a $13\ldots(p-1)2p$-match starting at position i also implies that $\sigma_i < \sigma_{i+p-2}$. It follows that $1,\ldots,\sigma_i-1$ must proceed σ_i which implies that $\sigma_i = i$. But since σ_{i+p-1} is greater than $\sigma_{i+1},\ldots,\sigma_{i+p-3}$, it follows that $\sigma_{i+p-2} = i+1$ and

$$\sigma_{i+1} = i+2,,\sigma_{i+3} = i+3,\ldots,\sigma_{i+p-3} = i+p-2.$$

Thus there is only one such σ which has $13\ldots(p-1)2p$-match starting at position i. As i can vary from 1 to $n-p+1$, it follows that there are $n-p+1$ permutations σ which starts with 1 and have 1 descent and contain a $13\ldots(p-1)2p$-match. Hence (23) holds.

4 Conclusions and some problems for further research.

We have proved that the polynomials $U_{1p23\ldots(p-1),n}(y)$ and $U_{134\ldots(p-1)2p,n}(y)$ satisfy simple recursions and that these recursions allow us to compute the initial terms in the generating functions $NM_{1p23\ldots(p-1)}(t,x,y)$ and $NM_{134\ldots(p-1)2p}(t,x,y)$ for $p \geq 4$.

It is easy to see that the polynomials $U_{1p23\ldots(p-1),n}(-y)$ and $U_{134\ldots(p-1)2p,n}(-y)$ are polynomials with non-negative integer coefficients. We have computed extensive tables of these polynomials and all the polynomials that we have computed are log-concave. Here a polynomial $P(y) = a_0 + a_1 y + \cdots + a_n y^n$ is called *log-concave* if for all $i = 2 \ldots n-1$, $a_{i-1}a_{i+1} < a_i^2$ and it is called *unimodal* if there exists an index k such that $a_i \leq a_{i+1}$ for $i = 1 \ldots k-1$ and $a_i \geq a_{i+1}$ for $i = k \ldots n-1$. Thus for any $p \geq 4$, we conjecture that the polynomials $U_{1p23\ldots(p-1),n}(-y)$ and $U_{134\ldots(p-1)2p,n}(-y)$ are log-concave.

We have computed $U_{n,\tau}(-y)$ for many permutations that start with 1. Out of all the patterns τ that start with 1 and have exactly one descent that we have looked at, all of the polynomials $U_{\tau,n}(-y)$ seem to be unimodal but not necessarily log-concave. For instance, the authors in [12] showed if $\tau = 1342$ then

$$U_{1342,n}(y) = (1-y)U_{1342,n-1}(y) - y\binom{n-2}{2}U_{1342,n-3}(y)$$

and the coefficients of y^i in $U_{1342,n}(-y)$ are given in Table 1.

Table 1: Coefficients of $U_{1342,n}(-y)$

Coefficients of y^i in $U_{1342,n}(-y)$										
	i=1	2	3	4	5	6	7	8	9	10
n=1	1									
2	1	1								
3	1	2	1							
4	1	4	3	1						
5	1	8	10	4	1					
6	1	15	30	20	5	1				
7	1	26	85	80	35	6	1			
8	1	42	231	315	175	56	7	1		
9	1	64	588	1176	910	336	84	8	1	
10	1	93	1380	4144	4326	2226	588	120	9	1

Notice that in row 8 and columns $6,7,8$, $(56)(1) > 7^2$. Hence, there are polynomials $U_{1342,n}(-y)$ that are not log-concave. Thus it would be interesting to see whether our recursions can be used to prove that the polynomials $U_{1p23\ldots(p-1),n}(-y)$ and $U_{134\ldots(p-1)2p,n}(-y)$ are log-concave.

If we set $y = 1$, then our results show that the $U_{1p2\ldots(p-1),n}(1)$ and the $U_{13\ldots(p-1)2p,n}(1)$ satisfy simple recursions. Nevertheless, it seems that the sequences $(U_{1p2\ldots(p-1),n}(1))_{n\geq 1}$ and $(U_{13\ldots(p-1)2p,n}(1))_{n\geq 1}$ are quite complicated. In fact, these sequences are not even monotone when we take absolute values. For example, the initial 27 terms of the sequence $(U_{15234,n}(1))_{n\geq 1}$ are

$$-1, 0, 0, 0, 1, 0, 0, -1, -5, 0, 1, 23, 45, -1, -82, -501, -584, 270, 3849, 12110, 9081, -25547, \\ -161741, -328989, -50941, 1784059, 6821610, \ldots$$

and the initial 27 terms of the sequence $(U_{1324,n}(1))_{n\geq 1}$ are

$$-1, 0, 0, 1, 0, -3, -1, 12, 6, -54, -33, 264, 181, -1365, -1008, 7345, 5712, -40713, -32890,$$
$$230886, 192045, -1333309, -1134912, 7813629, 6776639, -46351500, -40827423, \ldots.$$

A more general problem is to extend our method to the case of permutations that start with 1 but have more that one descent. The problem in this case is that the map I_τ is not an involution. That is, it is possible that when we split a brick b into two bricks b' and b'' at cell c labeled y, then it may be the case that b' can be combined with the brick b^- just before b because there is a descent between those bricks and there is no τ-match in the cells of b^- and b' while there was a τ-match in the cells of b^- and b so that we cannot combine b^- and b. Thus we can not use such a cell c to define an involution because we want the cases to be reversible. This means that we can not use such a cell c to define an involution so that we have to restrict ourselves to those cells c which are labeled with y where it is not possible to combine b^- and b'. This makes the definition of our involution more complicated and hence it is more difficult to analyze the fixed points of such involutions. Nevertheless, there is at least one special case where we can still carry out the analysis. Namely, the first author has shown $U_{15243,1}(y) = -y$ and for $n \geq 2$,

$$U_{15243,n}(y) = (1-y)U_{15243,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} y(-y)^{k-1}\binom{n-k-1}{k}U_{15243,n-2k}(y).$$

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Appendix: The polynomials $U_{1p23\ldots(p-1),n}(y)$, $NM_{1p23\ldots(p-1),n}(x,y)$, $U_{13\ldots(p-1)2p,n}(y)$, and $NM_{13\ldots(p-1)2p,n}(x,y)$.

Table 2: Coefficients of y^i in $U_{1423,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10
$n=1$	-1									
2	-1	1								
3	-1	2	-1							
4	-1	4	-3	1						
5	-1	7	-9	4	-1					
6	-1	11	-23	16	-5	1				
7	-1	16	-53	54	-25	6	-1			
8	-1	22	-110	165	-105	36	-7	1		

Table 3: The polynomials $NM_{1423,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 3\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 8\,x\,y^2 + 15\,x^2\,y^2 + 9\,x\,y^3 + 25\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 +$ $5\,x^2\,y^4 + 10\,x^3\,y^4 + 10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 20\,x\,y^2 + 31\,x^2\,y^2 + 46\,x\,y^3 + 113\,x^2\,y^3 + 90\,x^3\,y^3 + 23\,x\,y^4 + 79\,x^2\,y^4 +$ $105\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 + 15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 47\,x\,y^2 + 63\,x^2\,y^2 + 200\,x\,y^3 + 448\,x^2\,y^3 + 301\,x^3\,y^3 + 219\,x\,y^4 + 651\,x^2\,y^4 +$ $728\,x^3\,y^4 + 350\,x^4\,y^4 + 53\,x\,y^5 + 217\,x^2\,y^5 + 364\,x^3\,y^5 + 315\,x^4\,y^5 + 140\,x^5\,y^5 +$ $x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 + 21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 105\,x\,y^2 + 127\,x^2\,y^2 + 794\,x\,y^3 + 1650\,x^2\,y^3 + 966\,x^3\,y^3 + 1547\,x\,y^4 +$ $4225\,x^2\,y^4 + 4214\,x^3\,y^4 + 1701\,x^4\,y^4 + 919\,x\,y^5 + 3166\,x^2\,y^5 + 4410\,x^3\,y^5 +$ $3108\,x^4\,y^5 + 1050\,x^5\,y^5 + 115\,x\,y^6 + 543\,x^2\,y^6 + 1092\,x^3\,y^6 + 1204\,x^4\,y^6 +$ $770\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 + 70\,x^5\,y^7 +$ $56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$

Table 4: Coefficients of y^i in $U_{15234,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10	11
$n=1$	-1										
2	-1	1									
3	-1	2	-1								
4	-1	3	-3	1							
5	-1	5	-6	4	-1						
6	-1	8	-13	10	-5	1					
7	-1	12	-27	26	-15	6	-1				
8	-1	17	-52	65	-45	21	-7	1			

Table 5: The polynomials $NM_{15234,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 4\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 10\,x\,y^2 + 15\,x^2\,y^2 + 11\,x\,y^3 + 30\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 + 5\,x^2\,y^4 + 10\,x^3\,y^4 + 10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 23\,x\,y^2 + 31\,x^2\,y^2 + 63\,x\,y^3 + 140\,x^2\,y^3 + 90\,x^3\,y^3 + 26\,x\,y^4 + 91\,x^2\,y^4 + 120\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 + 15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 51\,x\,y^2 + 63\,x^2\,y^2 + 272\,x\,y^3 + 546\,x^2\,y^3 + 301\,x^3\,y^3 + 296\,x\,y^4 + 847\,x^2\,y^4 + 875\,x^3\,y^4 + 350\,x^4\,y^4 + 57\,x\,y^5 + 238\,x^2\,y^5 + 406\,x^3\,y^5 + 350\,x^4\,y^5 + 140\,x^5\,y^5 + x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 + 21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 110\,x\,y^2 + 127\,x^2\,y^2 + 1034\,x\,y^3 + 1948\,x^2\,y^3 + 966\,x^3\,y^3 + 2258\,x\,y^4 + 5746\,x^2\,y^4 + 5124\,x^3\,y^4 + 1701\,x^4\,y^4 + 1181\,x\,y^5 + 4048\,x^2\,y^5 + 5502\,x^3\,y^5 + 3640\,x^4\,y^5 + 1050\,x^5\,y^5 + 120\,x\,y^6 + 575\,x^2\,y^6 + 1176\,x^3\,y^6 + 1316\,x^4\,y^6 + 840\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 + 70\,x^5\,y^7 + 56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$

Table 6: Coefficients of y^i in $U_{162345,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10	11
$n=1$	-1										
2	-1	1									
3	-1	2	-1								
4	-1	3	-3	1							
5	-1	4	-6	4	-1						
6	-1	6	-10	10	-5	1					
7	-1	9	-18	20	-15	6	-1				
8	-1	13	-33	41	-35	21	-7	1			

Table 7: The polynomials $NM_{162345,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 4\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 11\,x\,y^2 + 15\,x^2\,y^2 + 11\,x\,y^3 + 30\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 + 5\,x^2\,y^4 +$ $10\,x^3\,y^4 + 10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 25\,x\,y^2 + 31\,x^2\,y^2 + 66\,x\,y^3 + 146\,x^2\,y^3 + 90\,x^3\,y^3 + 26\,x\,y^4 +$ $91\,x^2\,y^4 + 120\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 +$ $15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 54\,x\,y^2 + 63\,x^2\,y^2 + 298\,x\,y^3 + 581\,x^2\,y^3 + 301\,x^3\,y^3 + 302\,x\,y^4 +$ $868\,x^2\,y^4 + 896\,x^3\,y^4 + 350\,x^4\,y^4 + 57\,x\,y^5 + 238\,x^2\,y^5 + 406\,x^3\,y^5 +$ $350\,x^4\,y^5 + 140\,x^5\,y^5 + x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 +$ $21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 114\,x\,y^2 + 127\,x^2\,y^2 + 1151\,x\,y^3 + 2084\,x^2\,y^3 + 966\,x^3\,y^3 +$ $2406\,x\,y^4 + 6094\,x^2\,y^4 + 5348\,x^3\,y^4 + 1701\,x^4\,y^4 + 1191\,x\,y^5 + 4096\,x^2\,y^5 +$ $5586\,x^3\,y^5 + 3696\,x^4\,y^5 + 1050\,x^5\,y^5 + 120\,x\,y^6 + 575\,x^2\,y^6 + 1176\,x^3\,y^6 +$ $1316\,x^4\,y^6 + 840\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 +$ $70\,x^5\,y^7 + 56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$

Table 8: Coefficients of y^i in $U_{1324,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10	11
$n=1$	-1										
2	-1	1									
3	-1	2	-1								
4	-1	4	-3	1							
5	-1	6	-8	4	-1						
6	-1	8	-19	13	-5	1					
7	-1	10	-34	38	-19	6	-1				
8	-1	12	-53	98	-64	26	-7	1			

Table 9: The polynomials $NM_{1324,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 3\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 9\,x\,y^2 + 15\,x^2\,y^2 + 8\,x\,y^3 + 25\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 + 5\,x^2\,y^4 + 10\,x^3\,y^4 +$ $10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 23\,x\,y^2 + 31\,x^2\,y^2 + 48\,x\,y^3 + 119\,x^2\,y^3 + 90\,x^3\,y^3 + 20\,x\,y^4 + 73\,x^2\,y^4 +$ $105\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 + 15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 53\,x\,y^2 + 63\,x^2\,y^2 + 223\,x\,y^3 + 490\,x^2\,y^3 + 301\,x^3\,y^3 + 207\,x\,y^4 +$ $644\,x^2\,y^4 + 749\,x^3\,y^4 + 350\,x^4\,y^4 + 47\,x\,y^5 + 196\,x^2\,y^5 + 343\,x^3\,y^5 + 315\,x^4\,y^5 +$ $140\,x^5\,y^5 + x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 + 21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 115\,x\,y^2 + 127\,x^2\,y^2 + 925\,x\,y^3 + 1838\,x^2\,y^3 + 966\,x^3\,y^3 + 1602\,x\,y^4 +$ $4465\,x^2\,y^4 + 4466\,x^3\,y^4 + 1701\,x^4\,y^4 + 810\,x\,y^5 + 2930\,x^2\,y^5 + 4298\,x^3\,y^5 +$ $3164\,x^4\,y^5 + 1050\,x^5\,y^5 + 105\,x\,y^6 + 495\,x^2\,y^6 + 1008\,x^3\,y^6 + 1148\,x^4\,y^6 +$ $770\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 + 70\,x^5\,y^7 +$ $56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$

Table 10: Coefficients of y^i in $U_{13425,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10	11
$n=1$	-1										
2	-1	1									
3	-1	2	-1								
4	-1	3	-3	1							
5	-1	5	-6	4	-1						
6	-1	7	-12	10	-5	1					
7	-1	9	-21	23	-15	6	-1				
8	-1	11	-37	47	-39	21	-7	1			

Table 11: The polynomials $NM_{13425,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 4\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 10\,x\,y^2 + 15\,x^2\,y^2 + 11\,x\,y^3 + 30\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 + 5\,x^2\,y^4 +$ $10\,x^3\,y^4 + 10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 24\,x\,y^2 + 31\,x^2\,y^2 + 62\,x\,y^3 + 140\,x^2\,y^3 + 90\,x^3\,y^3 + 26\,x\,y^4 + 91\,x^2\,y^4 +$ $120\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 + 15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 54\,x\,y^2 + 63\,x^2\,y^2 + 273\,x\,y^3 + 553\,x^2\,y^3 + 301\,x^3\,y^3 + 292\,x\,y^4 +$ $840\,x^2\,y^4 + 875\,x^3\,y^4 + 350\,x^4\,y^4 + 57\,x\,y^5 + 238\,x^2\,y^5 + 406\,x^3\,y^5 + 350\,x^4\,y^5 +$ $140\,x^5\,y^5 + x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 + 21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 116\,x\,y^2 + 127\,x^2\,y^2 + 1071\,x\,y^3 + 2000\,x^2\,y^3 + 966\,x^3\,y^3 + 2228\,x\,y^4 +$ $5726\,x^2\,y^4 + 5152\,x^3\,y^4 + 1701\,x^4\,y^4 + 1171\,x\,y^5 + 4016\,x^2\,y^5 + 5474\,x^3\,y^5 +$ $3640\,x^4\,y^5 + 1050\,x^5\,y^5 + 120\,x\,y^6 + 575\,x^2\,y^6 + 1176\,x^3\,y^6 + 1316\,x^4\,y^6 +$ $840\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 + 70\,x^5\,y^7 +$ $56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$

Table 12: Coefficients of y^i in $U_{134526,n}(y)$

	$i=1$	2	3	4	5	6	7	8	9	10	11
$n=1$	-1										
2	-1	1									
3	-1	2	-1								
4	-1	3	-3	1							
5	-1	4	-6	4	-1						
6	-1	6	-10	10	-5	1					
7	-1	8	-17	20	-15	6	-1				
8	-1	10	-27	38	-35	21	-7	1			

Table 13: The polynomials $NM_{134526,n}(x,y)$

$n=1$	$x\,y$
2	$x\,y + x^2\,y^2$
3	$x\,y + x\,y^2 + 3\,x^2\,y^2 + x^3\,y^3$
4	$x\,y + 4\,x\,y^2 + 7\,x^2\,y^2 + x\,y^3 + 4\,x^2\,y^3 + 6\,x^3\,y^3 + x^4\,y^4$
5	$x\,y + 11\,x\,y^2 + 15\,x^2\,y^2 + 11\,x\,y^3 + 30\,x^2\,y^3 + 25\,x^3\,y^3 + x\,y^4 + 5\,x^2\,y^4 + 10\,x^3\,y^4 + 10\,x^4\,y^4 + x^5\,y^5$
6	$x\,y + 25\,x\,y^2 + 31\,x^2\,y^2 + 66\,x\,y^3 + 146\,x^2\,y^3 + 90\,x^3\,y^3 + 26\,x\,y^4 + 91\,x^2\,y^4 + 120\,x^3\,y^4 + 65\,x^4\,y^4 + x\,y^5 + 6\,x^2\,y^5 + 15\,x^3\,y^5 + 20\,x^4\,y^5 + 15\,x^5\,y^5 + x^6\,y^6$
7	$x\,y + 55\,x\,y^2 + 63\,x^2\,y^2 + 297\,x\,y^3 + 581\,x^2\,y^3 + 301\,x^3\,y^3 + 302\,x\,y^4 + 868\,x^2\,y^4 + 896\,x^3\,y^4 + 350\,x^4\,y^4 + 57\,x\,y^5 + 238\,x^2\,y^5 + 406\,x^3\,y^5 + 350\,x^4\,y^5 + 140\,x^5\,y^5 + x\,y^6 + 7\,x^2\,y^6 + 21\,x^3\,y^6 + 35\,x^4\,y^6 + 35\,x^5\,y^6 + 21\,x^6\,y^6 + x^7\,y^7$
8	$x\,y + 117\,x\,y^2 + 127\,x^2\,y^2 + 1153\,x\,y^3 + 2092\,x^2\,y^3 + 966\,x^3\,y^3 + 2401\,x\,y^4 + 6086\,x^2\,y^4 + 5348\,x^3\,y^4 + 1701\,x^4\,y^4 + 1191\,x\,y^5 + 4096\,x^2\,y^5 + 5586\,x^3\,y^5 + 3696\,x^4\,y^5 + 1050\,x^5\,y^5 + 120\,x\,y^6 + 575\,x^2\,y^6 + 1176\,x^3\,y^6 + 1316\,x^4\,y^6 + 840\,x^5\,y^6 + 266\,x^6\,y^6 + x\,y^7 + 8\,x^2\,y^7 + 28\,x^3\,y^7 + 56\,x^4\,y^7 + 70\,x^5\,y^7 + 56\,x^6\,y^7 + 28\,x^7\,y^7 + x^8\,y^8$