PU. M. A. Vol. **24** (2013), No.2, pp. 195 – 215

Up-down ascent sequences and the q-Genocchi numbers

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(Received: January 17, 2013, and in revised form July 15, 2013)

Abstract. Ascent sequences where introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [2] who showed that the number of ascent sequences of length n is the number of $(\mathbf{2}+\mathbf{2})$free posets of size n. We show that the Genocchi numbers can be interpreted as the number of up-down ascent sequences of odd length and the median Genocchi numbers can be interpreted as the number of up-down ascent sequences of even length. Moreover, we show how up-down ascent sequences can be used to interpret the q-analogues of the Seidel triangle for the Genocchi numbers defined by Zeng and Zhou [26] as well as a new bivariate analogue of the Seidel triangle for the Genocchi numbers.

Mathematics Subject Classification(2010). 05A15, 05A05.

Keywords: ascent sequence, Genocchi number, median Genocchi number, Seidel triangle, $2+2$-free poset, q-analogue.

1 Introduction

The Genocchi number G_{2n} for $n \geq 1$ [8, 23] can be defined through its relation with the Bernoulli numbers $G_{2n} = 2(2^{2n}-1)B_n$ or through its exponential generating function

$$\frac{2t}{e^t+1} = t + \sum_{n\geq 1}(-1)^n G_{2n}\frac{t^{2n}}{(2n)!}.$$

For example, $G_2 = 1$, $G_4 = 1$, $G_6 = 3$, $G_8 = 17$, $G_{10} = 155$, and $G_{12} = 2073$. The median Genocchi numbers H_{2n+1} are defined by

$$H_{2n+1} = \sum_{i\geq 0}(-1)^i G_{2n-2i}\binom{n}{2i+1}.$$

For example, $H_7 = 3G_6 - G_4 = 9-1 = 8$. The first few median Genocchi numbers are $H_1 = 1$, $H_3 = 1$, $H_5 = 2$, $H_7 = 8$, $H_9 = 56$, $H_{11} = 608$, and $H_{13} = 9440$.

Alternatively, one can define the Genocchi and median Genocchi numbers via the so-called Gandi generation [1] as

$$G_{2n+2} = \mathbb{B}_n(1) \text{ and } H_{2n+1} = \mathbb{C}_n(1) \text{ for } n \geq 1,$$

where $\mathbb{B}_n(x)$ and $\mathbb{C}_n(x)$ for $n \geq 1$ are the polynomials defined by

$$\begin{aligned}\mathbb{B}_1(x) &= x^2 \text{ and}\\ \mathbb{B}_n(x) &= x^2[\mathbb{B}_{n-1}(x+1) - \mathbb{B}_{n-1}(x)] \quad \text{for } n \geq 2\end{aligned}$$

and

$$\begin{aligned}\mathbb{C}_1(x) &= 1 \text{ and}\\ \mathbb{C}_n(x) &= (x+1)[(x+1)\mathbb{C}_{n-1}(x+1) - x\mathbb{C}_{n-1}(x)] \quad \text{for } n \geq 2.\end{aligned}$$

Seidel [22] gave a Pascal type triangle for the Genocchi numbers. That is, the Seidel triangle for the Genocchi numbers [11, 13, 25] is the array of integers $(g_{i,j})_{i,j\geq 1}$ where

$$\begin{aligned} g_{2i+1,j} &= g_{2i+1,j-1} + g_{2i,j}, \text{ for } j = 1,2,\ldots,i+1,\\ g_{2i,j} &= g_{2i,j+1} + g_{2i-1,j}, \text{ for } j = i, i-1, \ldots, 1,\end{aligned}$$

and $g_{i,j} = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$ by convention.

For example, the first values of $g_{i,j}$ for $1 \leq i,j \leq 10$ are given in the table below.

								155	155	5
						17	17	155	310	4
				3	3	17	34	138	448	3
		1	1	3	6	14	48	104	552	2
1	1	1	2	2	8	8	56	56	608	1
1	2	3	4	5	6	7	8	9	10	i/j

The table of the $g_{i,j}$s.

The Genocchi numbers and the median Genocchi numbers can then be defined as

$$G_{2n} = g_{2n-1,n} \text{ and } H_{2n-1} = g_{2n-1,1}$$

for all $n \geq 1$.

There are now many combinatorial interpretations of the Genocchi numbers. For example, Dumont [11] defined two sets of permutations now called Dumont permutations of type 1 and type 2 which are counted by the Genocchi numbers. That is, we say a permutation $\sigma = \sigma_1 \ldots \sigma_{2n}$ in the symmetric group S_{2n}

1. is a Dumont permutation of the first kind if

$$\begin{aligned}\sigma_i \text{ is even} &\Rightarrow i < 2n \text{ and } \sigma_i > \sigma_{i+1},\\ \sigma_i \text{ is odd} &\Rightarrow (i < 2n \text{ and } \sigma_i < \sigma_{i+1}) \text{ or } i = 2n \text{ and}\end{aligned}$$

2. is a Dumont permutation of the second kind if for every $i = 1, \ldots, n$,

$$\sigma_{2i} < 2i \text{ and } \sigma_{2i-1} \geq 2i - 1.$$

Let $\mathcal{D}_{2n}^{(i)}$ denote the set of Dumont permutations of type i in S_{2n} for $i = 1,2$. For example, $\mathcal{D}_2^{(1)} = \mathcal{D}_2^{(2)} = \{21\}$, $\mathcal{D}_4^{(1)} = \{2143, 3421, 4213\}$ and $\mathcal{D}_4^{(2)} = \{2143, 3142, 4132\}$. Then Dumont [11] showed that $|\mathcal{D}_{2n}^{(1)}| = |\mathcal{D}_{2n}^{(2)}| = G_{2n+2}$ for $n \geq 1$.

Solving a conjecture of Kitaev and Remmel [18, 19], Burstein, Josuat-Vergès, and Stromquist [5] proved that $|\mathcal{D}^{(3)}_{2n}| = G_{2n+2}$ where $\mathcal{D}^{(3)}_{2n}$ is the set of $\sigma = \sigma_1 \dots \sigma_{2n} \in S_{2n}$ whose descents only involve even numbers, i.e. $\sigma_i > \sigma_{i+1}$ implies σ_i and σ_{i+1} are even. Burstein, Josuat-Vergès, and Stromquist [5] also proved that $|\mathcal{D}^{(4)}_{2n}| = G_{2n+2}$ where $\mathcal{D}^{(4)}_{2n}$ is the set of $\sigma = \sigma_1 \dots \sigma_{2n} \in S_{2n}$ whose deficiencies only involve even number, i.e. $\sigma_i < i$ implies σ_i and i are even.

The elements of the Seidel triangle can also be interpreted in terms of these types of permutations. For example, let $\mathcal{G}^{(1)}_{2n,2k}$ denote the set of $\sigma \in \mathcal{D}^{(1)}_{2n}$ whose leftmost even letter is $2k$ and $\mathcal{H}^{(1)}_{2n,2k}$ denote the set of $\sigma \in \mathcal{D}^{(1)}_{2n}$ such that $\sigma_1 = 2k$ and the descending run starting at $2k$ ends in 1. Let $\mathcal{G}^{(3)}_{2n,2k}$ denote the set of $\sigma \in \mathcal{D}^{(3)}_{2n}$ whose leftmost even letter is $2k$ and $\mathcal{H}^{(3)}_{2n,2k}$ denote the set of $\sigma \in \mathcal{D}^{(3)}_{2n}$ such that $\sigma_2 = 2k$. Burstein and Stromquist [6] proved that

$$|\mathcal{H}^{(1)}_{2n,2k}| = g_{2n-1,n-k+1} \text{ and } |\mathcal{G}^{(1)}_{2n,2k}| = g_{2n,n-k+1}.$$

Later Burstein, Josuat-Vergès, and Stromquist [5] proved that

$$|\mathcal{H}^{(3)}_{2n,2k}| = g_{2n-1,n-k+1} \text{ and } |\mathcal{G}^{(3)}_{2n,2k}| = g_{2n,n-k+1}.$$

Dumont and Viennot [13] gave the first combinatorial interpretation to the elements in Seidel triangle via so-called alternating pistols. That is, an *alternating pistol* on $[m] = \{1, \dots, m\}$ is a mapping $p : [m] \to [m]$ such that for $i = 1, 2, \dots, \lceil m/2 \rceil$,
(1) $p(2i-1), p(2i) \leq i$ and
(2) $p(2i-1) \geq p(2i)$ and $p(2i) \leq p(2i+1)$.

A *strict alternating pistol* on $[m]$ is a mapping $p : [m] \to [m]$ such that for $i = 1, 2, \dots, \lceil m/2 \rceil$,
(1) $p(2i-1), p(2i) \leq i$ and
(2) $p(2i-1) \geq p(2i)$ and $p(2i) < p(2i+1)$.
We let $\mathcal{AP}_m$ and $\mathcal{SAP}_m$ on $[m]$.

Let $\mathcal{AP}_{i,j}$ ($\mathcal{SAP}_{i,j}$) be the set of alternating pistols p in $\mathcal{AP}_i$ (resp. strict alternating pistols in $\mathcal{SAP}_i$) such that $p(i) = j$. Then $\mathcal{AP}_i = \bigcup_{j \geq 1} \mathcal{AP}_{i,j}$ and $\mathcal{SAP}_i = \bigcup_{j \geq 1} \mathcal{SAP}_{i,j}$. Dumont and Viennot [13] proved that for all $i \geq 1$ and $1 \leq j \leq \lceil i/2 \rceil$,
(a) $g_{i,j} = |\mathcal{AP}_{i,j}|$,
(b) G_{2n} equals the number of alternating pistols on $[2n]$, and
(c) H_{2n+1} equals the number of strict alternating pistols on $[2n]$.

Zeng and Zhou [26] defined a q-analogue of the Seidel triangle by defining polynomials $(g_{i,j}(q))_{i,j\geq 1}$ where $g_{1,1}(q) = g_{2,1}(q) = 1$ and, for $i > 1$,

$$\begin{aligned} g_{2i+1,j}(q) &= g_{2i+1,j-1}(q) + q^{j-1} g_{2i,j}(q) \text{ for } j = 1, 2, \dots, i+1 \text{ and} & (1) \\ g_{2i,j}(q) &= g_{2i,j+1}(q) + q^{j-1} g_{2i-1,j}(q) \text{ for } j = 1, 2, \dots, i-1, i. & (2) \end{aligned}$$

Here $g_{i,j}(q) = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$ by convention.

For example, some initial values of $g_{i,j}(q)$ are given in the table below.

						$1+2q+3q^2+4q^3+4q^4+2q^5+q^6$	4
				$1+q+q^2$	$q^2+q^3+q^4$	$1+2q+4q^2+5q^3+4q^4+q^5$	3
		1	q	$1+q+q^2$	$q+2q^2+2q^3+q^4$	$1+2q+3q^2+4q^3+3q^4+q^5$	2
1	1	1	$1+q$	$1+q$	$1+2q+2q^2+2q^3+q^4$	$1+2q+2q^2+2q^3+q^4$	1
1	2	3	4	5	6	7	i/j

The table of the $g_{i,j}(q)$s.

They defined the q-Genocchi number $G_{2n}(q)$ and the median q-Genocchi number $H_{2n-1}(q)$ by

$$G_{2n}(q) = g_{2n-1,n}(q) \text{ and } H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q).$$

The *charge*, $ch(p)$, of an alternating pistol $p : [m] \to [m]$ is defined to be $ch(p) = \sum_{i=1}^{m}(p(i)-1)$. Zeng and Zhou [26] proved that

$$g_{i,j}(q) = \sum_{p \in \mathcal{AP}_{i,j}} q^{ch(p)-j+1}.$$

In addition, they proved that

$$G_{2n+2}(q) = \sum_{p \in AP_{2n}} q^{ch(p)} \text{ and } H_{2n+1}(q) = \sum_{p \in SAP_{2n}} q^{ch(p)}.$$

The main goal of this paper is to give a new combinatorial interpretation of the elements of the q-analogue of the Seidel triangle in terms of q-counting a certain subset of ascent sequences. Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [2] to study the problem of enumerating $(\mathbf{2}+\mathbf{2})$-free posets. We will show that the polynomials $g_{i,j}(q)$ can easily be defined in terms of up-down ascent sequences. In particular, we shall show that G_{2n} is the number of up-down ascent sequences of length $2n-1$ and that H_{2n-1} is the number of up-down ascent sequences of length $2n-2$. Moreover, we shall show that one can use up-down ascent sequences to define a 2-variable refinement $g_{i,j}(q,z)$ of the Seidel triangle such that $g_{i,j}(q) = g_{i,j}(q,1)$.

Let $\mathbb{N} = \{0, 1, \ldots, \}$ denote the natural numbers. A sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ is an *ascent sequence of length* n if and only if it satisfies $a_1 = 0$ and $a_i \in [0, 1 + \text{asc}(a_1, \ldots, a_{i-1})]$ for all $2 \leq i \leq n$. Here, for any integer sequence $(a_1, \ldots, a_i)$, the number of *ascents* of this sequence is

$$\text{asc}(a_1, \ldots, a_i) = |\{j : a_j < a_{j+1}\}|.$$

For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence which has 4 ascents. We let Asc denote the set of all ascent sequences where we assume the empty word is also an ascent sequence. For any $n \geq 1$, we let Asc_n denote the set of all ascent sequences of length n. If $a = a_1 \ldots a_n \in Asc_n$, we let $|a| = n$ be the length of a, $\sum a = a_1 + \cdots + a_n$ equal the sum of the values of a, $\text{zero}(a)$ denote the number of occurrences of 0 in a, and $\ell(a) = a_n$ denote the last letter of a.

We say that $a = a_1 \ldots a_n \in Asc_n$ is an *up-down* ascent sequence if $a_1 < a_2 > a_3 < a_4 > \cdots$. That is, $a = a_1 \ldots a_n \in Asc_n$ is an up-down ascent sequence if $a_i < a_{i+1}$ whenever i is odd and $a_i > a_{i+1}$ whenever i is even. Let UDA_n equal the set of up-down ascent sequences of length n. For example, $UDA_1 = \{0\}$, $UDA_2 = \{01\}$, $UDA_3 = \{010\}$, $UDA_4 = \{0102, 0101\}$,
$UDA_5 = \{01020, 01021, 01010\}$, etc. Let $UDA_n^{(i)}$ denote the set of elements $a = a_1 \ldots a_n \in UDA_n$ such that $a_n = i$. If $n \geq 1$ and $a = a_1 \ldots a_n \in UDA_n$, then we define the weight of a, $w(a)$, by

$$w(a) = \sum_{i=1}^{n-1}(a_i - \chi(i \text{ is even})) = \left(\sum a\right) - a_n - \lfloor (n-1)/2 \rfloor \tag{3}$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Then the main result of this paper is the following theorem.

THEOREM 1.1 *For all* $1 \le j \le \lceil i/2 \rceil$,

$$\begin{aligned} g_{2i,j}(q) &= \sum_{a=a_1\dots a_{2i+1}\in UDA_{2i+1}^{(j-1)}} q^{w(a)} \text{ and} \\ g_{2i+1,j}(q) &= \sum_{a=a_1\dots a_{2i+2}\in UDA_{2i+2}^{(j)}} q^{w(a)}. \end{aligned}$$

It follows for all $n \ge 1$,

$$G_{2n}(q) = g_{2n-1,n}(q) = \sum_{\substack{a=a_1\dots a_{2n}\in UDA_{2n} \\ a_{2n}=n}} q^{w(a)}. \tag{4}$$

Note that the condition that $a_{2n} = n$ imposes no restriction on a_{2n-1} so that

$$G_{2n} = |UDA_{2n-1}|. \tag{5}$$

Similarly, for $n \ge 1$,

$$H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q) = q^{n-2} \sum_{\substack{a=a_1\dots a_{2n}\in UDA_{2n} \\ a_{2n}=1}} q^{w(a)}. \tag{6}$$

Note that if $a_{2n} = 1$, then $a_{2n-1} = 0$. But then the condition that $a_{2n-1} = 0$ imposes no restriction on a_{2n-2} so that

$$H_{2n-1} = |UDA_{2n-2}|. \tag{7}$$

In fact, we can define a natural 2-variable refinement of the Seidel triangle by defining

$$\begin{aligned} \tilde{g}_{2i,j}(q,z) &:= \sum_{a=a_1\dots a_{2i+1}\in UDA_{2i+1}^{(j-1)}} q^{w(a)} z^{\text{zero}(a)} \text{ and} \\ \tilde{g}_{2i+1,j}(q) &:= \sum_{a=a_1\dots a_{2i+2}\in UDA_{2i+2}^{(j)}} q^{w(a)} z^{\text{zero}(a)}. \end{aligned}$$

Then we shall show that $\tilde{g}_{1,1}(q,z) = z$ and $\tilde{g}_{2,1}(q,z) = z^2$ and, for $i > 1$,

$$\begin{aligned} \tilde{g}_{2i+1,j}(q,z) &= \tilde{g}_{2i+1,j-1}(q,z) + q^{j-1}\tilde{g}_{2i,j}(q,z) \text{ for } j = 1,\dots,i+1, \\ \tilde{g}_{2i,j}(q,z) &= \tilde{g}_{2i,j+1}(q,z) + q^{j-1}\tilde{g}_{2i-1,j}(q,z) \text{ for } j = 2,\dots,i, \text{ and} \\ \tilde{g}_{2i,1}(q,z) &= z\tilde{g}_{2i,2}(q,z) + z\tilde{g}_{2i-1,1}(q,z). \end{aligned}$$

We note that there has been considerable work on ascent sequences in recent years, see [2, 9, 15, 20]. Moreover, ascent sequences are important because they are in bijection with several other interesting combinatorial objects. That is, it follows from the work of [2, 10, 7] that there are natural bijections between Asc_n and the following four classes of combinatorial objects.

(1) The set of $(2+2)$-free posets of size n. Here we consider two posets to be equal if they are isomorphic, and an unlabeled poset is said to be $(2+2)$-free if it does not contain an induced subposet that is isomorphic to $(2+2)$, the union of two disjoint 2-element chains.

(2) The set M_n of upper triangular matrices of nonnegative integers such that no row or column contains all zero entries, and the sum of the entries is n.

(3) The set R_n of permutations of $[n]$ where, in each occurrence of the pattern 231, either the letters corresponding to the 2 and the 3 are nonadjacent, or else the letters corresponding to the 2 and the 1 are nonadjacent in value.

(4) The set Mch_n of Stoimenow matchings on $[2n]$. A *matching* of the set $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into subsets of size 2, each of which is called an *arc*. The smaller number in an arc is its *opener* and the larger one its *closer*. A matching is said to be *Stoimenow* if it has no pair of arcs $\{a < b\}$ and $\{c < d\}$ that satisfy one (or both) of the following conditions: (a) $a = c+1$ and $b < d$ and (b) $a < c$ and $b = d+1$. In other words, a Stoimenow matching has no pair of arcs such that one is nested within the other and either the openers or the closers of the two arcs differ by 1.

It is possible to find the images of up-down ascent sequences under the bijections of [2, 10, 7] to find combinatorial interpretations of the Genocchi and median Genocchi numbers in each of these settings. However, the corresponding conditions in each of these settings are not as natural as up-down ascent sequences so we will not pursue these conditions in this paper.

The outline of this paper is the following. In section 2, we shall give two proofs of Theorem 1.1. Our first proof will show that the $g_{i,j}(q,z)$'s satisfy the recursions given above. Our second proof uses a simple weight-preserving bijection between up-down ascent sequences and alternating pistols. In section 3, we shall consider variations of the up-down condition on ascent sequences. That is, we say that $a = a_1 \ldots a_n \in Asc_n$ is a *strict-up-weak-down* ascent sequence if $a_1 < a_2 \geq a_3 < a_4 \geq \cdots$. Thus, $a = a_1 \ldots a_n \in Asc_n$ is a strict-up-weak-down ascent sequence if $a_i < a_{i+1}$ whenever i is odd and $a_i \geq a_{i+1}$ whenever i is even. We let $SUWDA_n$ denote the set of weak-up-strict-down ascent sequences. We say that $a = a_1 \ldots a_n \in Asc_n$ is a *weak-up-weak-down* ascent sequence if $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots$. Thus, $a = a_1 \ldots a_n \in Asc_n$ is a weak-up-weak-down ascent sequence if $a_i \leq a_{i+1}$ whenever i is odd and $a_i \geq a_{i+1}$ whenever i is even. We let $WUWDA_n$ denote the set of weak-up-weak-down ascent sequences. We say that $a = a_1 \ldots a_n \in Asc_n$ is a *weak-up-strict-down* ascent sequence if $a_1 \leq a_2 > a_3 \leq a_4 > \cdots$. Thus, $a = a_1 \ldots a_n \in Asc_n$ is a weak-up-strict-down ascent sequence if $a_i \leq a_{i+1}$ whenever i is odd and $a_i > a_{i+1}$ whenever i is even. We let $WUSDA_n$ denote the set of weak-up-strict-down ascent sequences. We shall study the following generating functions:

$$\begin{aligned} A_n(q,x,y,z) &= \sum_{a \in UDA_n} q^{\sum a} x^{|a|-1} y^{\ell(a)} z^{\mathrm{zero}(a)}, \\ B_n(q,x,y,z) &= \sum_{a \in SUWDA_n} q^{\sum a} x^{|a|-1} y^{\ell(a)} z^{\mathrm{zero}(a)}, \end{aligned}$$

$$\begin{aligned} C_n(q,u,x,y,z) &= \sum_{a \in WUWDA_n} q^{\sum a} u^{\mathrm{asc}(a)} x^{|a|} y^{\ell(a)} z^{\mathrm{zero}(a)}, \text{ and} \\ D_n(q,u,x,y,z) &= \sum_{a \in WUSDA_n} q^{\sum a} u^{\mathrm{asc}(a)} x^{|a|} y^{\ell(a)} z^{\mathrm{zero}(a)}. \end{aligned}$$

We shall show that these generating functions statisfy some simple recursions and we shall study some of the properties of the sets UDA_n, $SUWDA_n$, $WUSDA_n$, and $WUWDA_n$. For example, we shall prove that for all $n \geq 1$, $|SUWDA_n| = |UDA_{n+1}|$ and that $|WUWD_n| \leq |UDA_{n+2}|$.

2 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. In fact, we give two proofs. Our first proof is to verify directly that our combinatorial interpretation of the $g_{i,j}(q)$s satisfy the required recursions. We give it first because it shows that the recursions are quite natural in the setting of ascent sequences. Our second proof will be to give a simple weight preserving bijection between up-down ascent sequences and alternating pistols.

In fact, our first proof allows us to give a refinement of the q-analogue of the Seidel triangle described above. That is, let

$$\begin{aligned} \tilde{g}_{2i,j}(q,z) &:= \sum_{a=a_1 \ldots a_{2i+1} \in UDA_{2i+1}^{(j-1)}} q^{w(a)} z^{\mathrm{zero}(a)} \text{ and} \\ \tilde{g}_{2i+1,j}(q,z) &:= \sum_{a=a_1 \ldots a_{2i+2} \in UDA_{2i+2}^{(j)}} q^{w(a)} z^{\mathrm{zero}(a)}. \end{aligned}$$

Then we shall show that $\tilde{g}_{1,1}(q,z) = z$ and $\tilde{g}_{2,1}(q,z) = z^2$ and

$$\tilde{g}_{2i+1,j}(q,z) = \tilde{g}_{2i+1,j-1}(q,z) + q^{j-1}\tilde{g}_{2i,j}(q,z) \text{ for } j = 1,\ldots,i+1, \tag{8}$$

$$\tilde{g}_{2i,j}(q,z) = \tilde{g}_{2i,j+1}(q,z) + q^{j-1}\tilde{g}_{2i-1,j}(q,z) \text{ for } j = 2,\ldots,i, \tag{9}$$

and

$$\tilde{g}_{2i,1}(q,z) = z\tilde{g}_{2i,2}(q,z) + z\tilde{g}_{2i-1,1}(q,z). \tag{10}$$

Clearly when $z = 1$, these initial conditions and recursions are the same as the initial conditions and recursions satisfied by the $g_{i,j}(q)$s so that it will follow that $\tilde{g}_{i,j}(q,1) = g_{i,j}(q)$ for all i and j.

Since $UDA_2^{(1)} = \{01\}$ and $w(01) = 0$, we have that

$$\tilde{g}_{1,1}(q,z) = \sum_{a \in UDA_2^{(1)}} q^{w(a)} z^{\mathrm{zero}(a)} = z.$$

Similarly, since $UDA_3^{(0)} = \{010\}$ and $w(010) = 0$, we have that

$$\tilde{g}_{2,1}(q) = \sum_{a \in UDA_3^{(0)}} q^{w(a)} z^{\mathrm{zero}(a)} = z^2.$$

Next we prove (8) in the case where $j \neq i+1$ by proving that for $i \geq 2$ and $1 \leq j \leq i$,

$$q^{j-1}\tilde{g}_{2i,j}(q,z) = \tilde{g}_{2i+1,j}(q,z) - \tilde{g}_{2i+1,j-1}(q,z). \tag{11}$$

Now

$$\tilde{g}_{2i+1,j}(q,z) - \tilde{g}_{2i+1,j-1}(q,z) = \sum_{a \in UDA_{2i+2}^{(j)}} q^{w(a)} z^{\text{zero}(a)} - \sum_{b \in UDA_{2i+2}^{(j-1)}} q^{w(b)} z^{\text{zero}(b)}.$$

Now suppose that $a = a_1 \dots a_{2i} a_{2i+1} j \in UDA_{2i+2}^{(j)}$. Then we know that $0 \leq a_{2i+1} < j$. Partitioning the elements of $UDA_{2i+2}^{(j)}$ according to the second to last letter, we see that

$$UDA_{2i+2}^{(j)} = \bigcup_{s=0}^{j-1} UDA_{2i+2}^{(j,s)}$$

where $UDA_{2i+2}^{(j,s)} = \{a_1 \dots a_{2i+2} \in UDA_{2i+2} : a_{2i+1} = s \text{ and } a_{2i+2} = j\}$. Similarly,

$$UDA_{2i+2}^{(j-1)} = \bigcup_{s=0}^{j-2} UDA_{2i+2}^{(j-1,s)}$$

where $UDA_{2i+2}^{(j-1,s)} = \{a_1 \dots a_{2i+2} \in UDA_{2i+2} : a_{2i+1} = s \text{ and } a_{2i+2} = j-1\}$. Note that since we are assuming that $1 \leq j \leq i$, it follows that if $s < j-1$, then

$$a_1 \dots a_{2i} s j \in UDA_{2i+2}^{(j,s)} \Leftrightarrow a_1 \dots a_{2i} s(j-1) \in UDA_{2i+2}^{(j-1,s)}$$

and

$$\begin{aligned} w(a_1 \dots a_{2i} s j) &= s + \sum_{i=1}^{2i} (a_i - \chi(i \text{ is even})) = w(a_1 \dots a_{2i} s(j-1)) \text{ and} \\ \text{zero}(a_1 \dots a_{2i} s j) &= \text{zero}(a_1 \dots a_{2i} s(j-1)). \end{aligned}$$

Hence, for $0 \leq s < j-1$,

$$\sum_{a \in UDA_{2i+2}^{(j,s)}} q^{w(a)} z^{\text{zero}(a)} = \sum_{a \in UDA_{2i+2}^{(j-1,s)}} q^{w(a)} z^{\text{zero}(a)}.$$

Thus it follows that if $1 \leq j \leq i$,

$$\tilde{g}_{2i+1,j}(q,z) - \tilde{g}_{2i+1,j-1}(q,z) = \sum_{a \in UDA_{2i+2}^{(j,j-1)}} q^{w(a)} z^{\text{zero}(a)}. \tag{12}$$

But then

$$a_1 \dots a_{2i}(j-1)j \in UDA_{2i+2}^{(j,j-1)} \iff a_1 \dots a_{2i}(j-1) \in UDA_{2i+1}^{(j-1)}$$

and

$$\begin{aligned} w(a_1 \dots a_{2i}(j-1)j) &= j-1+\sum_{i=1}^{2i}(a_i - \chi(i \text{ is even})) = j-1+w(a_1 \dots a_{2i}(j-1)) \text{ and} \\ \text{zero}(a_1 \dots a_{2i}(j-1)j) &= \text{zero}(a_1 \dots a_{2i}(j-1)). \end{aligned}$$

Thus

$$\sum_{a \in UDA_{2i+2}^{(j,j-1)}} q^{w(a)} z^{\text{zero}(a)} = q^{j-1} \sum_{b \in UDA_{2i+1}^{(j-1)}} q^{w(b)} z^{\text{zero}(b)} = q^{j-1} \tilde{g}_{2i,j}(q,z).$$

This shows that

$$\tilde{g}_{2i+1,j}(q,z) = \tilde{g}_{2i+1,j-1}(q,z) + q^{j-1}\tilde{g}_{2i,j}(q,z) \tag{13}$$

for $1 \leq j \leq i$.

To complete our proof of (8), we must prove that

$$\tilde{g}_{2i+1,i+1}(q,z) = \tilde{g}_{2i+1,i}(q,z) + q^i \tilde{g}_{2i,i+1}(q,z) \tag{14}$$

But note that

$$\tilde{g}_{2i,i+1}(q,z) = \sum_{a=a_1 \dots a_{2i+1} \in UDA_{2i+1}^{(i)}} q^{w(a)} z^{\text{zero}(a)}.$$

However, $a_{2i} \leq i$ since $a_1 \dots a_{2i-1}$ has exactly $i-1$ ascents which means that $a_{2i+1} < a_{2i} \leq i$. Thus $UDA_{2i+1}^{(i)} = \emptyset$ and $\tilde{g}_{2i+1,i+1}(q,z) = 0$. Thus to prove (14), we must show that

$$\begin{aligned} \tilde{g}_{2i+1,i+1}(q,z) &= \sum_{a=a_1 \dots a_{2i+2} \in UDA_{2i+2}^{(i+1)}} q^{w(a)} z^{\text{zero}(a)} \\ &= \sum_{a=a_1 \dots a_{2i+2} \in UDA_{2i+2}^{(i)}} q^{w(a)} z^{\text{zero}(a)} = \tilde{g}_{2i+1,i}(q,z). \end{aligned}$$

However, if $a_1 \dots a_{2i+1}(i+1) \in UDA_{2i+2}^{(i+1)}$, we know that $a_{2i+1} \leq i-1$ which means that $a_1 \dots a_{2i+1} i \in UDA_{2i+2}^{(i)}$. Similarly, if $a_1 \dots a_{2i+1} i \in UDA_{2i+2}^{(i)}$, then there are i ascents in $a_1 \dots a_{2i+1}$ so that $a_1 \dots a_{2i+1}(i+1) \in UDA_{2i+2}^{(i+1)}$. Since the last letter of an up-down ascent sequence does not contribute to its weight and we are assuming that $i \geq 2$, it follows that

$$\sum_{a=a_1 \dots a_{2i+2} \in UDA_{2i+2}^{(i+1)}} q^{w(a)} z^{\text{zero}(a)} = \sum_{a=a_1 \dots a_{2i+2} \in UDA_{2i+2}^{(i)}} q^{w(a)} z^{\text{zero}(a)}.$$

Hence $\tilde{g}_{2i+1,i+1}(q,z) = \tilde{g}_{2i+1,i}(q,z)$ as desired. This completes our proof of (8).

Next we prove (9) by proving that for $i \geq 2$ and $2 \leq j \leq i$,

$$q^{j-1}\tilde{g}_{2i-1,j}(q,z) = \tilde{g}_{2i,j}(q,z) - \tilde{g}_{2i,j+1}(q,z). \tag{15}$$

Now

$$\tilde{g}_{2i,j}(q,z) - \tilde{g}_{2i,j+1}(q,z) = \sum_{a \in UDA_{2i+1}^{(j-1)}} q^{w(a)} z^{\text{zero}(a)} - \sum_{b \in UDA_{2i+1}^{(j)}} q^{w(b)} z^{\text{zero}(b)}.$$

Now suppose that $a = a_1 \ldots a_{2i}(j-1) \in UDA_{2i+1}^{(j-1)}$. First observe that $a_1 \ldots a_{2i-1}$ has exactly $i-1$ ascents since $a_1 \ldots a_{2i-1}$ is an up-down ascent sequence. It follows that the maximum value of a_{2i} is i. Hence if $a = a_1 \ldots a_{2i}(j-1) \in UDA_{2i+1}^{(j-1)}$, then we must have that $j-1 < a_{2i} \leq i$. Partitioning the elements of $UDA_{2i+1}^{(j-1)}$ according to the second to last letter, we see that

$$UDA_{2i+1}^{(j-1)} = \bigcup_{s=j}^{i} UDA_{2i+1}^{(j-1,s)}$$

where $UDA_{2i+1}^{(j-1,s)} = \{a_1 \ldots a_{2i+1} \in UDA_{2i+1} : a_{2i} = s \text{ and } a_{2i+1} = (j-1)\}$. Similarly,

$$UDA_{2i+1}^{(j)} = \bigcup_{s=j+1}^{i} UDA_{2i+1}^{(j,s)}$$

where $UDA_{2i+1}^{(j,s)} = \{a_1 \ldots a_{2i+1} \in UDA_{2i+1} : a_{2i} = s \text{ and } a_{2i+1} = j\}$. Now suppose that $j < s \leq i$. Then since $j \geq 2$,

$$a_1 \ldots a_{2i-1}s(j-1) \in UDA_{2i+1}^{(j-1,s)} \Leftrightarrow a_1 \ldots a_{2i-1}sj \in UDA_{2i+1}^{(j,s)}$$

and

$$\begin{aligned} w(a_1 \ldots a_{2i-1}s(j-1)) &= s-1+\sum_{i=1}^{2i-1}(a_i - \chi(i \text{ is even})) = w(a_1 \ldots a_{2i-1}sj) \text{ and} \\ \text{zero}(a_1 \ldots a_{2i-1}s(j-1)) &= \text{zero}(a_1 \ldots a_{2i-1}sj). \end{aligned}$$

Hence

$$\sum_{a=a_1 \ldots a_{2i-1}s(j-1) \in UDA_{2i+1}^{(j-1,s)}} q^{w(a)} z^{\text{zero}(a)} = \sum_{a=a_1 \ldots a_{2i-1}sj \in UDA_{2i+1}^{(j,s)}} q^{w(a)} z^{\text{zero}(a)}.$$

It follows that

$$\tilde{g}_{2i,j}(q,z) - \tilde{g}_{2i,j+1}(q,z) = \sum_{a \in UDA_{2i+1}^{(j-1,j)}} q^{w(a)} z^{\text{zero}(a)}. \tag{16}$$

But then

$$a_1 \ldots a_{2i-1}j(j-1) \in UDA_{2i+1}^{(j-1,j)} \iff a_1 \ldots a_{2i-1}j \in UDA_{2i}^{(j)}$$

and

$$\begin{aligned} w(a_1 \ldots a_{2i-1}j(j-1)) &= j-1+\sum_{i=1}^{2i-1}(a_i - \chi(i \text{ is even})) = j-1+w(a_1 \ldots a_{2i}j) \text{ and} \\ \text{zero}(a_1 \ldots a_{2i-1}j(j-1)) &= \text{zero}(a_1 \ldots a_{2i}j). \end{aligned}$$

so that

$$\sum_{a \in UDA_{2i+1}^{(j-1,j)}} q^{w(a)} z^{\text{zero}(a)} = q^{j-1} \sum_{b \in UDA_{2i}^{(j)}} q^{w(b)} z^{\text{zero}(b)} = q^{j-1}\tilde{g}_{2i-1,j}(q,z)$$

which proves (9).

Finally, we must prove (10) in the case where $i \geq 2$. Now

$$\tilde{g}_{2i,1}(q) - z\tilde{g}_{2i,2}(q) = \sum_{a \in UDA_{2i+1}^{(0)}} q^{w(a)} z^{\text{zero}(a)} - z \sum_{b \in UDA_{2i+1}^{(1)}} q^{w(b)} z^{\text{zero}(b)}.$$

Then as above, we can write

$$UDA_{2i+1}^{(0)} = \bigcup_{s=1}^{i} UDA_{2i+1}^{(0,s)}$$

and

$$UDA_{2i+1}^{(1)} = \bigcup_{s=2}^{i} UDA_{2i+1}^{(1,s)}.$$

Now suppose that $1 < s \leq i$. Then

$$a_1 \ldots a_{2i-1}s0 \in UDA_{2i+1}^{(0,s)} \Leftrightarrow a_1 \ldots a_{2i-1}s1 \in UDA_{2i+1}^{(1,s)}$$

and

$$w(a_1 \ldots a_{2i-1}s0) = s - 1 + \sum_{i=1}^{2i-1}(a_i - \chi(i \text{ is even})) = w(a_1 \ldots a_{2i-1}s1).$$

Thus

$$\sum_{a=a_1 \ldots a_{2i-1}s0 \in UDA_{2i+1}^{(0,s)}} q^{w(a)} z^{\text{zero}(a)} = z \sum_{a=a_1 \ldots a_{2i-1}s1 \in UDA_{2i+1}^{(1,s)}} q^{w(a)} z^{\text{zero}(a)}.$$

It follows that

$$\tilde{g}_{2i,1}(q,z) - z\tilde{g}_{2i,2}(q,z) = \sum_{a \in UDA_{2i+1}^{(0,1)}} q^{w(a)} z^{\text{zero}(a)}. \tag{17}$$

But then

$$a_1 \ldots a_{2i-1}10 \in UDA_{2i+1}^{(0,1)} \iff a_1 \ldots a_{2i-1}1 \in UDA_{2i}^{(1)}$$

and

$$w(a_1 \ldots a_{2i-1}10) = \sum_{i=1}^{2i-1}(a_i - \chi(i \text{ is even})) = w(a_1 \ldots a_{2i-1}1).$$

Hence

$$\sum_{a \in UDA_{2i+1}^{(0,1)}} q^{w(a)} z^{\text{zero}(a)} = z \sum_{b \in UDA_{2i}^{(1)}} q^{w(b)} z^{\text{zero}(b)} = z\tilde{g}_{2i-1,1}(q,z)$$

which proves (10).

The tables below give some initial values of $\tilde{g}_{i,j}(q)$.

				$z^3+qz^3+q^2z^2$	$q^2z^3+q^3z^3+q^4z^2$	3
		z^2	qz^2	$z^3+qz^3+q^2z^2$	$qz^3+2q^2z^3+q^3(z^2+z^3)+q^4z^2$	2
z	z^2	z^2	z^3+qz^3	z^3+qz^3	$z^4+2qz^4+2q^2z^4+q^3(z^3+z^4)+q^4z^3$	1
1	2	3	4	5	6	i/j

$z^4+2qz^4+q^2(z^3+2z^4)+q^3(3z^3+z^4)+q^4(3z^3+z^4)+q^5(z^2+z^3)+q^6z^2$	4
$z^4+2qz^4+q^2(z^3+2z^4)+q^3(3z^3+z^4)+q^4(3z^3+z^4)+q^5(z^2+z^3)+q^6z^2$	3
$z^4+2qz^4+q^2z^3+2z^4)+q^3(3z^3+z^4)+q^4(2z^3+z^4)+q^5z^2$	2
$z^4+2qz^4+2q^2z^4+q^3(z^3+z^4)+q^4z^3$	1
7	i/j

The table of the $g_{i,j}(q,z)$s.

Our second proof of Theorem 1.1 is a bijective proof suggested to us by Alex Burstein. That is, suppose that we are given an up-down sequence $a = a_1a_2\dots a_n$ where $n \geq 2$. Then let $\theta(a) = b_1\dots b_{n-1}$ where

$$b_i = \begin{cases} a_{i+1} & \text{if } i+1 \text{ is even and} \\ a_{i+1}+1 & \text{if } i+1 \text{ is odd.} \end{cases} \tag{18}$$

That is, $b_1\dots b_{n-1}$ arises from $a_1a_2\dots a_n$ by first removing a_1 and then adding 1 to a_j if j is odd. Thus for example, $\theta(010213130) = 11223231$.

First we show that θ is a bijection from ascent sequences onto alternating pistols. That is, it is easy to see that for all $s \geq 1$, $a_{2s} > a_{2s+1}$ if and only if $b_{2s-1} \geq b_{2s}$ and $a_{2s+1} < a_{2s+2}$ if and only if $b_{2s} \leq b_{2s+1}$. Moreover, if $a = a_1\dots a_n$ is an ascent sequence, then for all $s \geq 1$ such that $2s \leq n$, $s \geq a_{2s} > a_{2s-1}$ so that $s \geq b_{2s-1} \geq b_{2s}$. Thus if a is an ascent sequence of length n, then $\theta(a)$ is an alternating pistol of length $n-1$. Vice versa, if $b_1\dots b_n$ is an alternating pistol of length n, then $\theta^{-1}(b_1\dots b_n) = 0b_1(b_2-1)b_3(b_4-1)\dots$. It follows that $\theta^{-1}(b_1\dots b_n)$ is an up-down sequence that starts with 0 and $a_{2s} = b_{2s-1} \leq s$ for all $s \geq 1$. It then easily follows that for all i $a_i \leq 1 + \operatorname{asc} a_1\dots a_{i-1}$. Thus $\theta^{-1}(b_1\dots b_n)$ is always an up-down ascent sequence.

We need only verify that θ shows that

$$g_{2i,j}(q) = \sum_{p \in \mathcal{AP}_{2i,j}} q^{ch(p)-j+1} = \sum_{a \in UDA^{(j-1)}_{2i+1}} q^{w(a)}. \tag{19}$$

and

$$g_{2i+1,j}(q) = \sum_{p \in \mathcal{AP}_{2i+1,j}} q^{ch(p)-j+1} = \sum_{a \in UDA^{(j)}_{2i+2}} q^{w(a)}. \tag{20}$$

For (19), note that if $a = a_1a_2\dots a_{2i}a_{2i+1} \in UDA^{(j-1)}_{2i+1}$, then

$$\theta(a) = a_2(a_3+1)a_4(a_5+1)\dots a_{2i}(a_{2i+1}+1) = b_1\dots b_{2i}.$$

Since $a_{2i+1} = j-1$, then $b_{2i} = a_{2i+1}+1 = j$ so that $\theta(a) \in \mathcal{AP}_{2i,j}$. Moreover,

$$\begin{aligned} ch(b_1\dots b_{2i}) &= (b_1-1)+(b_2-1)+(b_3-1)+(b_4-1)+\dots+(b_{2i-1}-1)+(b_{2i}-1)-j+1 \\ &= (b_1-1)+(b_2-1)+(b_3-1)+(b_4-1)+\dots+(b_{2i-1}-1) \\ &= (a_2-1)+a_3+(a_4-1)+a_5+\dots+(a_{2i}-1) = w(a). \end{aligned}$$

Thus θ shows that (19) holds.

For (20), note that if $a = a_1a_2 \dots a_{2i}a_{2i+2} \in UDA^{(j)}_{2i+2}$, then

$$\theta(a) = a_2(a_3+1)a_4(a_5+1)\dots a_{2i}(a_{2i+1}+1)a_{2i+2} = b_1 \dots b_{2i+1}.$$

Since $a_{2i+2} = b_{2i+1} = j$, it follows that $\theta(a) \in \mathcal{AP}_{2i+1,j}$. Moreover,

$$\begin{aligned} ch(b_1 \dots b_{2i}b_{2i+1}) &= (b_1-1)+(b_2-1)+\cdots+(b_{2i}-1)+(b_{2i+1}-1)-j+1 \\ &= (b_1-1)+(b_2-1)+\cdots+(b_{2i}-1) \\ &= (a_2-1)+a_3+\cdots+(a_{2i}-1)+a_{2i+1} = w(a). \end{aligned}$$

Thus θ shows that (20) holds.

3 Alternative up-down conditions on ascent sequences.

In this section, we shall study certain natural generating functions on the sets UDA_n, $WUSDA_n$, $SUWDA_n$, and $WUWDA_n$. Our first goal is to prove some simple recursions for these generating functions. We start by considering up-down sequences and strict-up-weak-down sequences which are sequences where we know exactly where the ascents in the sequences occur. For any $n \geq 1$, let

$$A_n(q,x,y,z) := \sum_{a \in UDA_n} q^{\sum a} x^{|a|-1} y^{\ell(a)} z^{\text{zero}(a)} \tag{21}$$

and

$$B_n(q,x,y,z) := \sum_{a \in SUWDA_n} q^{\sum a} x^{|a|-1} y^{\ell(a)} z^{\text{zero}(a)}. \tag{22}$$

Then each term $q^k x^{2n} y^\ell z^m$ that appears in $A_{2n+1}(q,x,y,z)$ corresponds to an up-down ascent sequence $a = a_1 \dots a_{2n+1}$ such that $\sum a = k$, $\text{zero}(a) = m$, and $a_{2n+1} = \ell$. Each such sequence gives rise to sequences in UDA_{2n+2} of the form $a_1 \dots a_{2n+1}a_{2n+2}$ where $a_{2n+2} \in \{\ell+1,\dots,n+1\}$. That is, we are forced to have $a_{2n+1} < a_{2n+2}$. Moreover, in an up-down ascent sequence $a = a_1 \dots a_{2n+1} \in UDA_{2n+1}$, $\text{asc}(a_1 \dots a_{2n+1}) = n$ so that $a_{2n+2} \leq n+1$. Thus each term $q^k x^{2n} z^m y^\ell$ contributes a factor of $q^k x^{2n+1} z^m((qy)^{\ell+1}+\cdots+(qy)^{n+1})$ to $A_{2n+2}(q,x,y,z)$. Note that

$$\begin{aligned} q^k x^{2n+1} z^m((qy)^{\ell+1}+\cdots+(qy)^{n+1}) &= q^k x^{2n+1} z^m (qy)^{\ell+1}(1+(qy)+\cdots+(qy)^{n-\ell}) \\ &= q^k x^{2n+1} z^m (qy)^{\ell+1} \frac{1-(qy)^{n-\ell+1}}{1-qy} \\ &= \frac{x}{1-qy} q^k x^{2n} z^m((qy)^{\ell+1}-(qy)^{n+2}) \\ &= \frac{x}{1-qy}(qyq^k x^{2n} z^m (qy)^\ell - q^2y^2q^k z^m (xq^{1/2}y^{1/2})^{2n}). \end{aligned}$$

It follows that

$$A_{2n+2}(q,x,y,z) = \frac{x}{1-qy}(qyA_{2n+1}(q,x,qy,z) - q^2y^2A_{2n+1}(q,xq^{1/2}y^{1/2},1,z)). \tag{23}$$

The same reasoning will show that

$$B_{2n+2}(q,x,y,z) = \frac{x}{1-qy}(qyB_{2n+1}(q,x,qy,z) - q^2y^2B_{2n+1}(q,xq^{1/2}y^{1/2},1,z)). \tag{24}$$

Next each term $q^k x^{2n-1} y^\ell z^m$ that appears in $A_{2n}(q,x,y,z)$ corresponds to an up-down ascent sequence $a = a_1 \ldots a_{2n}$ such that $\sum a = k$, $\text{zero}(a) = m$, and $a_{2n} = \ell$. Each such sequence gives rise to sequences in UDA_{2n+1} of the form $a_1 \ldots a_{2n}a_{2n+1}$ where $a_{2n+1} \in \{0,\ldots,\ell-1\}$ since we must have $a_{2n} > a_{2n+1}$. Thus each such term $q^k x^{2n-1} z^m y^\ell$ contributes a factor of

$$q^k x^{2n} z^m (z + (qy) + \cdots + (qy)^{\ell-1}) = x(z-1)q^k x^{2n-1} z^m + q^k x^{2n} z^m (1 + (qy) + \cdots + (qy)^{\ell-1})$$

to $A_{2n+1}(q,x,y)$. Note that

$$\begin{aligned} q^k x^{2n} z^m (1 + (qy) + \cdots + (qy)^{\ell-1}) &= q^k x^{2n} z^m \frac{1-(qy)^\ell}{1-qy} \\ &= \frac{x}{1-qy}(q^k x^{2n-1} z^m - q^k x^{2n-1} z^m (qy)^\ell). \end{aligned}$$

It follows that

$$A_{2n+1}(q,x,y,z) = \frac{x}{1-qy}(A_{2n}(q,x,1,z) - A_{2n}(q,x,qy,z)) + x(z-1)A_{2n}(q,x,1,z). \tag{25}$$

Similarly each term $q^k x^{2n-1} y^\ell z^m$ that appears in $B_{2n}(q,x,y,z)$ corresponds to a strict-up-weak-down ascent sequence $a = a_1 \ldots a_{2n}$ such that $\sum a = k$, $\text{zero}(a) = m$, and $a_{2n} = \ell$. Each such sequence gives rise to sequences in $SUWDA_{2n+1}$ of the form $a_1 \ldots a_{2n}a_{2n+1}$ where $a_{2n+1} \in \{0,\ldots,\ell\}$ since we must have $a_{2n} \geq a_{2n+1}$. Thus each such term $q^k x^{2n-1} z^m y^\ell$ contributes a factor of

$$q^k x^{2n} z^m (z + (qy) + \cdots + (qy)^\ell) = q^k x^{2n} z^m (1 + (qy) + \cdots + (qy)^\ell) + x(z-1)q^k x^{2n-1} z^m$$

to $B_{2n+1}(q,x,y,z)$. Note that

$$\begin{aligned} q^k x^{2n} z^m (1 + (qy) + \cdots + (qy)^\ell) &= q^k x^{2n} z^m \frac{1-(qy)^{\ell+1}}{1-qy} \\ &= \frac{x}{1-qy}(q^k x^{2n-1} z^m - qyq^k x^{2n-1} z^m (qy)^\ell). \end{aligned}$$

It follows that

$$B_{2n+1}(q,x,y,z) = \frac{x}{1-qy}(B_{2n}(q,x,1,z) - qyB_{2n}(q,x,qy,z)) + x(z-1)B_{2n}(q,x,1,z). \tag{26}$$

Note that $UDA_1 = \{0\}$, $UDA_2 = \{01\}$, and $UDA_3 = \{010\}$. One can use Mathematica to iterate the recursions (23) and (25) to compute some initial terms in the sequence $(A_n(q,x,y,z))_{n\geq 1}$. For example, we have computed the following table.

n	$A_n(q,x,y,z)$
1	z
2	$x(qyz)$
3	$x^2(qz^2)$
4	$x^3(q^2yz^2+q^3y^2z^2)$
5	$x^4(q^2z^3+q^3z^3+q^4yz^2)$
6	$x^5\left(q^6y^2z^2+q^7y^3z^2+q^3yz^3+q^4yz^3+q^4y^2z^3+q^5y^2z^3+q^5y^3z^3+q^6y^3z^3\right)$
7	$x^6\left(q^7yz^2+q^8yz^2+q^9y^2z^2+q^6z^3+q^7z^3+q^5yz^3+2q^6yz^3+q^7yz^3+q^7y^2z^3+ q^8y^2z^3+q^3z^4+2q^4z^4+2q^5z^4+q^6x^6z^4\right)$

Similarly $SUWDA_1=\{0\}$, $SUWDA_2=\{01\}$, and $SUWDA_3=\{010,011\}$. One can use Mathematica to iterate the recursions (24) and (26) to compute some initial terms in the sequence $(B_n(q,x,y,z))_{n\geq 1}$. For example, we have computed the following table.

n	$B_n(q,x,y,z)$
1	z
2	$x(qyz)$
3	$x^2(qz^2+q^2yz)$
4	$x^3(q^4y^2z+q^2yz^2+q^3y^2z^2)$
5	$x^4(q^5yz+q^6y^2z+q^4z^2+q^3yz^2+q^4yz^2+q^5y^2z^2+q^2z^3+q^3z^3)$
6	$x^5\left(q^7y^2z+q^8y^3z+q^9y^3z+q^5yz^2+q^5y^2z^2+2q^6y^2z^2+q^6y^3z^2+ 2q^7y^3z^2+q^8y^3z^2+q^3yz^3+q^4yz^3+q^4y^2z^3+q^5y^2z^3+q^5y^3z^3+q^6y^3z^3\right)$
7	$x^6\left(q^8yz+q^9yz+q^{10}yz+q^9y^2z+q^{10}y^2z+q^{11}y^2z+q^{11}y^3z+q^{12}y^3z+ q^7z^2+q^8z^2+q^9z^2+2q^6yz^2+3q^7yz^2+2q^8yz^2+q^9yz^2+q^7y^2z^2+ 3q^8y^2z^2+2q^9y^2z^2+q^{10}y^2z^2+q^9y^3z^2+2q^{10}y^3z^2+q^{11}y^3z^2+2q^5z^3+ 3q^6z^3+2q^7z^3+q^8z^3+q^4yz^3+2q^5yz^3+2q^6yz^3+q^7yz^3+q^6y^2z^3+ 2q^7y^2z^3+q^8y^2z^3+q^8y^3z^3+q^9y^3z^3+q^3z^4+2q^4z^4+2q^5z^4+q^6z^4\right)$

Note that $A_n(1,1,1,1)=|UDA_n|$ and $B_n(1,1,1,1)=|SUWDA_n|$ for all $n\geq 1$. Thus our recursions also allow us to compute the following.

1. The initial terms of the sequence $(|UDA_n|)_{n\geq 1}$ are

$$1,1,1,2,3,8,17,56,155,608,2073,\ldots.$$

2. The initial terms of the sequence $(|SUWDA_n|)_{n\geq 1}$ are

$$1,1,2,3,8,17,56,155,608,2073,9440,\ldots.$$

Comparing these two sequences, one is naturally led to conjecture that for $n\geq 1$,

$$|SUWD_n|=|UDA_{n+1}|. \tag{27}$$

In fact, there is a simple bijection which proves (27). That is, we define a bijection $\Gamma_n: UDA_{n+1}\to SUWDA_n$ for all $n\geq 1$. Given $a_1a_2\ldots a_{n+1}\in UDA_{n+1}$, we know that for all $i\geq 1$, $a_{2i}\leq i$ since

$\text{asc}(a_1 \dots a_{2i-1}) = i-1$. Thus $a_{2i+1} < a_{2i} \leq i$. We then let $\Gamma_n(a_1 \dots a_{n+1}) = b_1 b_2 \dots b_n$ where for $i \geq 1$, $b_{2i-1} = i - a_{2i}$ and $b_{2i} = i - a_{2i+1}$. It follows that $b_j \geq 0$ for all j. For example, if $a = 0102120$, then

$$\Gamma_7(a) = (1-1)(1-0)(2-2)(2-1)(3-2)(3-0) = 010113.$$

Note that $a_2 = 1$ so that $b_1 = 1 - a_2 = 0$ as required. Similarly, since $a_{2i+1} < a_{2i}$, it follows that $b_{2i-1} < b_{2i} \leq i$ for all i. Also $a_{2i+1} < a_{2i+2}$ which implies that $b_{2i} = (i - a_{2i+1}) \leq (i+1-a_{2i+2}) = b_{2i+1}$. Thus $\Gamma(a_1 \dots a_{n+1})$ is always an element of $SUWD_n$. Similarly, we can define $\Gamma_{n+1}^{-1}(b_1 \dots b_n) = 0a_2 \dots a_{n+1}$ where $a_{2i} = i - b_{2i-1}$ and $a_{2i+1} = i - b_{2i}$.

For the generating functions for $WUWD_n$ and $WUSD_n$, we must keep track of more information since the positions of the ascents in such sequences are not all the same. To this end, define

$$C_n(q,u,x,y,z) := \sum_{a \in WUWDA_n} q^{\sum a} u^{\text{asc}(a)} x^{|a|} y^{\ell(a)} z^{\text{zero}(a)} \tag{28}$$

and

$$D_n(q,u,x,y,z) := \sum_{a \in WUSDA_n} q^{\sum a} u^{\text{asc}(a)} x^{|a|} y^{\ell(a)} z^{\text{zero}(a)}. \tag{29}$$

Each term $q^k u^s x^{2n+1} z^m y^\ell$ that appears in $C_{2n+1}(q,u,x,y,z)$ corresponds to a weak-up-weak-down ascent sequence $a = a_1 \dots a_{2n+1}$ such that $\sum a = k$, $\text{asc}(a) = s$, $\text{zero}(a) = m$, and $a_{2n+1} = \ell$. Each such sequence gives rise to sequences in $WUWDA_{2n+2}$ of the form $a_1 \dots a_{2n+1} a_{2n+2}$ where $a_{2n+2} \in \{\ell, \dots, s+1\}$. Now if $a_{2n+2} = \ell$, then we do not increase the number of ascents so that such sequences contribute a factor of $q^k u^s x^{2n+2} z^m (qy)^\ell$ to $C_{2n+2}(q,u,x,y,z)$ if $\ell > 0$ and $q^k u^s x^{2n+2} z^{m+1}$ to $C_{2n+2}(q,u,x,y,z)$ if $\ell = 0$. Now $C_{2n+1}(q,u,x,0,z)$ is the generating function for the sequences $a = a_1 \dots a_{2n+1} \in SUSDA_{2n+1}$ such that $a_{2n+1} = 0$ and $C_{2n+1}(q,u,x,y,z) - C_{2n+1}(q,u,x,0,z)$ is the generating function for the sequences $a = a_1 \dots a_{2n+1} \in SUSDA_{2n+1}$ such that $a_{2n+1} > 0$. Thus the sequences $a = a_1 \dots a_{2n+1} a_{2n+2} \in SUSDA_{2n+2}$ such that $a_{2n+1} = a_{2n+2}$ contribute a factor

$$zxC_{2n+1}(q,u,x,0,z) + x(C_{2n+1}(q,u,x,qy,z) - C_{2n+1}(q,u,x,0,z))$$

to $C_{2n+2}(q,u,x,y,z)$. The sequences where $\ell + 1 \leq a_{2n+2} \leq s+1$ contribute a factor

$$q^k u^{s+1} x^{2n+2} z^m ((qy)^{\ell+1} + \dots + (qy)^{s+1})$$

to $C_{2n+2}(q,u,x,y,z)$. Note that

$$\begin{aligned}
&q^k u^{s+1} x^{2n+2} z^m ((qy)^{\ell+1} + \dots + (qy)^{s+1}) = \\
&q^k u^{s+1} x^{2n+2} z^m (qy)^{\ell+1} (1 + (qy) + \dots + (qy)^{s-\ell}) = \\
&q^k u^{s+1} x^{2n+2} z^m (qy)^{\ell+1} \frac{1-(qy)^{s-\ell+1}}{1-qy} = \\
&\frac{x}{1-qy} q^k u^{s+1} x^{2n+1} z^m ((qy)^{\ell+1} - (qy)^{s+2}) = \\
&\frac{x}{1-qy} (uqyq^k u^s x^{2n+1} z^m (qy)^\ell - uq^2y^2 q^k z^m (uqy)^s x^{2n+1}).
\end{aligned}$$

It follows that

$$C_{2n+2}(q,u,x,y,z) = zxC_{2n+1}(q,u,x,0,z)+x(C_{2n+1}(q,u,x,qy,z)-C_{2n+1}(q,u,x,0,z))+ \frac{x}{1-qy}(uqyC_{2n+1}(q,u,x,qy,z)-uq^2y^2C_{2n+1}(q,uqy,x,1,z)). \quad (30)$$

The same reasoning will show that

$$D_{2n+2}(q,u,x,y,z) = zxD_{2n+1}(q,u,x,0,z)+x(D_{2n+1}(q,u,x,qy,z)-D_{2n+1}(q,u,x,0,z))+ \frac{x}{1-qy}(uqyD_{2n+1}(q,u,x,qy,z)-uq^2y^2D_{2n+1}(q,uqy,x,1,z)). \quad (31)$$

The same reasoning that we used to prove (26) will prove that

$$C_{2n+1}(q,u,x,y,z) = \frac{x}{1-qy}(C_{2n}(q,u,x,1,z)-qyC_{2n}(q,u,x,qy,z))+x(z-1)C_{2n}(q,u,x,1,z). \quad (32)$$

However, we must be a bit more careful in the case of $D_{2n+1}(q,u,x,y,z)$. That is, because we allow only weak rises at odd positions, it is possible to have a weak-up-strict-down ascent sequence $a=a_1\dots a_{2n+1}$ where $a_{2n+1}=0$. For example, the sequence $(01)^{n-1}00$ is always in $WUSD_{2n}$. However, unlike the other cases, this sequence cannot be extended to a sequence in $WUSD_{2n+1}$ since we cannot decrease from 0. Fortunately, the sequences $WUSD_{2n}$ that end in 0 correspond to the terms in $D_{2n}(q,u,x,0,z)$. Now each term $q^ku^sx^{2n}z^my^\ell$ that appears in $D_{2n}(q,u,x,y,z)-D_{2n}(q,u,x,0,z)$ corresponds to weak-up-strict-down ascent sequences $a=a_1\dots a_{2n}$ such that $0<a_{2n}=\ell$. Each such sequence gives rise to sequences in $WUSDA_{2n+1}$ of the form $a_1\dots a_{2n}a_{2n+1}$ where $a_{2n+1}\in\{0,\dots,\ell-1\}$ since we must have $a_{2n}>a_{2n+1}$. Thus each such term $q^ku^sx^{2n}z^my^\ell$ contributes a factor of

$$q^ku^sx^{2n+1}z^m(z+(qy)+\cdots+(qy)^{\ell-1}) = q^ku^sx^{2n+1}z^m(1+(qy)+\cdots+(qy)^{\ell-1})+x(z-1)q^ku^sx^{2n}z^m$$

to $D_{2n+1}(q,u,x,y,z)$. Note that

$$\begin{aligned} q^ku^sx^{2n+1}z^m(1+(qy)+\cdots+(qy)^{\ell-1}) &= q^ku^sx^{2n+1}z^m\frac{1-(qy)^\ell}{1-qy} \\ &= \frac{x}{1-qy}(q^ku^sx^{2n}z^m-q^ku^sx^{2n}z^m(qy)^\ell). \end{aligned}$$

It follows that

$$\begin{aligned} & D_{2n+1}(q,u,x,y) = \quad (33) \\ & \frac{x}{1-qy}((D_{2n}(q,u,x,1,z)-D_{2n}(q,u,x,0,z))-(D_{2n}(q,u,x,qy,z)-D_{2n}(q,u,x,0,z)))+ \\ & x(z-1)(D_{2n}(q,u,x,1,z)-D_{2n}(q,u,x,0,z)) = \\ & \frac{x}{1-qy}(D_{2n}(q,u,x,1,z)-D_{2n}(q,u,x,qy,z))+ \\ & x(z-1)(D_{2n}(q,u,x,1,z)-D_{2n}(q,u,x,0,z)). \end{aligned}$$

Note that $WUWDA_1=\{0\}$, $WUWDA_2=\{00,01\}$, and $WUWDA_3=\{000,010,011\}$. One can use Mathematica to iterate the recursions (30) and (32) to compute some initial terms in the sequence $(C_n(q,u,x,y,z))_{n\geq 1}$. For example, we have computed the following table.

n	$C_n(q,u,x,y,z)$
1	$x(z)$
2	$x^2(z^2+quyz)$
3	$x^3(z^3+quz^2+q^2uyz)$
4	$x^4(q^3uyz+q^4u^2y^2z+q^2u^2yz^2+q^3u^2y^2z^2+quz^3+quyz^3+z^4)$
5	$x^5\left(q^4uyz+q^5u^2yz+q^6u^2y^2z+q^3uz^2+q^4u^2z^2+q^3u^2yz^2+q^4u^2yz^2+ q^5u^2x^5y^2z^2+q^2u^2z^3+q^3u^2z^3+q^2uyz^3+2quz^4+z^5\right)$
6	$x^6\left(1+2qu+q^3u+q^2u^2+q^3u^2+q^4u^2+quy+q^3uy+q^5uy+2q^2u^2y+2q^4u^2y+ q^5u^2y+q^6u^2y+q^3u^3y+q^4u^3y+q^5u^3y+2q^3u^2y^2+q^5uyz+q^6u^2yz+ q^6u^2y^2z+q^8u^2y^2z+q^7u^3y^2z+q^8u^3y^3z+q^9u^3y^3z+2q^4u^2yz^2+q^5u^2yz^2+ q^5u^3yz^2+q^5u^2y^2z^2+q^7u^2y^2z^2+q^5u^3y^2z^2+2q^6u^3y^2z^2+q^6u^3y^3z^2+ 2q^7u^3y^3z^2+q^8u^3y^3z^2+q^3uz^3+q^4u^2z^3+q^3uyz^3+q^3u^3yz^3+q^4u^3yz^3+ q^4u^2y^2z^3+q^4u^3y^2z^3+q^5u^3y^2z^3+q^5u^3y^3z^3+q^6u^3y^3z^3+q^2u^2z^4+q^3u^2z^4+ 2q^2u^2yz^4+2q^3u^2y^2z^4+2quz^5+quyz^5+z^6\right)$

Similarly $WUSDA_1=\{0\}$, $WUSD_2=\{00,01\}$, and $WUSDA_3=\{010\}$. One can use Mathematica to iterate recursions (31) and (33) to compute some initial terms in the sequence $(D_n(q,u,x,y,z))_{n\geq 1}$. For example, we have computed the following table.

n	$D_n(q,u,x,y,z)$
1	$x(z)$
2	$x^2(z^2+quyz)$
3	$x^3(quz^2)$
4	$x^4(q^2u^2yz^2+q^3u^2y^2z^2+quz^3)$
5	$x^5(q^4u^2yz^2+q^2u^2z^3+q^3u^2z^3)$
6	$x^6\left(q^5u^2yz^2+q^6u^3y^2z^2+q^7u^3y^3z^2+q^3u^3yz^3+q^4u^3yz^3+q^4u^3y^2z^3+ q^5u^3y^2z^3+q^5u^3y^3z^3+q^6u^3y^3z^3+q^2u^2z^4+q^3u^2z^4\right)$

Note that $C_n(1,1,1,1,1)=|WUWDA_n|$ and $D_n(1,1,1,1,1)=|WUSA_n|$ for all $n\geq 1$. Thus our recursions also allow us to compute the following.

1. The initial terms of the sequence $(|WUWDA_n|)_{n\geq 1}$ are
$$1,2,3,7,14,40,102,354,1113,4576,17210,\ldots.$$
2. The initial terms of the sequence $(|WUSDA_n|)_{n\geq 1}$ are
$$1,2,1,3,3,11,18,77,173,846,2419\ldots.$$

Neither of these sequences nor their odd entries nor their even entries appear in the On-line Encyclopedia of Integer Sequences (OEIS).

It is obvious that $UDA_n\subseteq WUSDA_n$ and $SUWD_n\subseteq WUWDA_n$ so that
$$|UDA_n|\leq|WUSDA_n| \text{ and } |SUWDA_n|\leq|WUWDA_n|.$$

It is also the case that for all $n\geq 1$, $|WUWDA_n|\leq|UDA_{n+2}|$. That is, for all $n\geq 1$, we define a injection $\theta: WUWDA_n\rightarrow UDA_{n+2}$ by defining $\theta(a_1\ldots a_n)=01b_1\ldots b_n$ where $b_i=a_i$ if i is odd and

$b_i = a_i + 1$ if i is even. It is obvious that $\theta(a_1 \ldots a_n)$ satisfies the up-down condition. However, we must verify that $\theta(a_1 \ldots a_n)$ satisfies the conditions to be an ascent sequence. Note that a_1 must be equal to zero so that $b_1 = 0$ and the sequence $\theta(a_1 \ldots a_n)$ starts out with 010. It is easy to see that $01a_1 \ldots a_{2i}$ is an ascent sequence. Moreover, the number of ascents of $01a_1 \ldots a_{2i}$ is one more than the number of ascents of $a_1 \ldots a_{2i}$. Since $a_{2i} \leq \text{asc}(a_1 \ldots a_{2i-1})$, it follows that

$$a_{2i} + 1 \leq \text{asc}(01a_1 \ldots a_{2i-1}) \leq \text{asc}(01b_1 \ldots b_{2i-1}).$$

Similarly, it is easy to see that

$$a_{2i+1} \leq \text{asc}(a_1 \ldots a_{2i-1}) \leq \text{asc}(01b_1 \ldots b_{2i-1}).$$

Thus $\theta(a_1 \ldots a_n)$ is always an ascent sequence.

Finally, we observe that one can easily iterate our pairs of recursions to get functional equations for the generating functions of even and odd terms in our generating functions. For example, iterating (23) and (25), we see that

$$\begin{aligned} A_{2n+2}(q,x,y,z) &= \frac{qyx^2}{(1-qy)(1-q^2y)}(A_{2n}(q,x,1,z) - A_{2n}(q,x,q^2y,z)) - \\ &\quad \frac{(qy)^{5/2}x^2}{(1-q)(1-qy)}(A_{2n}(q,x(qy)^{1/2},1,z) - A_{2n}(q,x(qy)^{1/2},q,z)) + \\ &\quad \frac{qyx^2(z-1)}{1-qy}(A_{2n}(q,x,1,z) - qyA_{2n}(q,x(qy)^{1/2},1,z)). \end{aligned} \tag{34}$$

Thus if we let $A^{(2)}(q,x,y,z) = \sum_{n\geq 1} A_{2n}(q,x,y,z)$, then we see that

$$\begin{aligned} A^{(2)}(q,x,y,z) &= xqyz + \frac{qyx^2}{(1-qy)(1-q^2y)}(A^{(2)}(q,x,1,z) - A^{(2)}(q,x,q^2y,z)) - \\ &\quad \frac{(qy)^{5/2}x^2}{(1-q)(1-qy)}(A^{(2)}(q,x(qy)^{1/2},1,z) - A^{(2)}(q,x(qy)^{1/2},q)) + \\ &\quad \frac{qyx^2(z-1)}{1-qy}(A^{(2)}(q,x,1,z) - qyA^{(2)}(q,x(qy)^{1/2},1,z)). \end{aligned} \tag{35}$$

Similarly, by iterating (23) and (25), one can show that

$$\begin{aligned} A_{2n+1}(q,x,y,z) &= \frac{qx^2(z+qy-qyz}{(1-q)(1-qy)}(A_{2n-1}(q,x,q,z) - A_{2n-1}(q,xq^{1/2},1,z)) - \\ &\quad \frac{q^2yx^2}{(1-qy)(1-q^2y)}(A_{2n-1}(q,x,q^2y,z) - A_{2n-1}(q,x(qy)^{1/2},1,z)). \end{aligned} \tag{36}$$

Thus if we let $A^{(1)}(q,x,y,z) = \sum_{n\geq 1} A_{2n-1}(q,x,y,z)$, then we see that

$$\begin{aligned} A^{(1)}(q,x,y,z) &= z + \frac{qx^2(z+qy-qyz)}{(1-q)(1-qy)}(A^{(1)}(q,x,q,z) - A^{(1)}(q,xq^{1/2},1,z)) - \\ &\quad \frac{q^2yx^2}{(1-qy)(1-q^2y)}(A^{(1)}(q,x,q^2y,z) - A^{(1)}(q,x(qy)^{1/2},1,z)). \end{aligned} \tag{37}$$

However, neither (35) nor (37) appear to be particularly useful in finding closed forms for infinite series expansions of $A^{(2)}(q,x,y)$ or $A^{(1)}(q,x,y)$. The functional equations in the other cases are similar.

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